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A question analogous to the flux conjecture concerning Lagrangian submanifolds

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Abstract. We discuss a question analogous to the flux conjecture concerning the space of Lagrangian submanifolds. Under suitable conditions, we prove C^1-closedness of orbits of the action of Hamiltonian diffeomorphism group on the space of Lagrangian submanifolds.

1. Introduction

In this note, we discuss a question analogous to the flux conjecture for the quotient space of Lagrangian submanifolds modulo the Hamiltonian diffeomorphism group of the ambient symplectic manifold and related questions. We briefly recall the flux conjecture, see section 2 for more details. Let (M, ω) be a closed symplectic manifold and $\mathrm{Ham}(M, \omega)$ the group of Hamiltonian diffeomorphisms of (M, ω). It is obviously a subgroup in the identity component $\mathrm{Symp}_0(M, \omega)$ of the group of symplectomorphisms. The C^1-flux conjecture states that $\mathrm{Ham}(M, \omega)$ is closed $\mathrm{Symp}_0(M, \omega)$ with respect to the C^1-topology. This conjecture is equivalent to that the so-called flux group Γ_ω is discrete in $\mathrm{H}^1(M; \mathbf{R})$. From now on, we simply call it the flux conjecture. We proved the flux conjecture in [5] using Floer cohomology for symplectic isotopies (Floer-Novikov theory). Note that it implies that the space $\mathrm{Symp}_0(M, \omega)/\mathrm{Ham}(M, \omega) \cong \mathrm{H}^1(M; \mathbf{R})/\Gamma_\omega$ is Hausdorff.

For a symplectomorphism ϕ of (M, ω), its graph $Gr_\phi \subset M \times M$ is a Lagrangian submanifold with respect to the symplectic form $-\pi_1^*\omega + \pi_2^*\omega$. Hence the statement on symplectomorphisms can be rephrased in terms of associated Lagrangian submanifolds, namely their graphs. We can also consider a Lagrangian submanifold L in a general symplectic manifold (P, Ω). This leads to the following question.

Let L be a closed embedded Lagrangian submanifold in a closed symplectic submanifold (P, Ω). Denote by $\mathrm{Lag}(L)$ the space of all embedded Lagrangian submanifolds, which are Lagrangian isotopic to L. The group $\mathrm{Ham}(P, \Omega)$ of Hamiltonian diffeomorphisms acts on $\mathrm{Lag}(L)$ in a natural way.

Question 1.1. Is $\mathrm{Lag}(L)/\mathrm{Ham}(P, \Omega)$ Hausdorff with respect to the C^1-topology?

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In general, the answer is negative. Chekanov [1] studied a certain class of embedded Lagrangian tori in symplectic vector spaces. The Hausdorffness property fails at monotone Lagrangian tori in his examples. Although the question is known to be false in general, we would still like to discuss the question under appropriate conditions on the Lagrangian submanifold L. In this note, we mainly discuss the following weaker question.

Question 1.2. Are $\mathrm{Ham}(P,\Omega)$-orbits closed in $\mathrm{Lag}(L)$ with respect to the C^1-topology?

It is natural to expect the relation between the stability for Lagrangian submanifolds and the above question, cf. [8], [9] as well as [3]. Our main result is the following:

Theorem 1.1. *Let (P,Ω) be a closed symplectic manifold and $L \subset P$ a closed embedded Lagrangian submanifold with vanishing Maslov class. If L is unobstructed in the sense of* [4]*, $\mathrm{Ham}(P,\Omega)\cdot L$ is closed in $\mathrm{Lag}(L)$ with respect to the C^1-topology.*

Corollary 1.2. *Let (P,Ω) be a closed symplectic manifold and $L \subset P$ a closed embedded Lagrangian submanifold with vanishing Maslov class. If $\mathrm{H}^2(L;\mathbb{Q}) = 0$, $\mathrm{Ham}(P,\Omega)\cdot L$ is closed in $\mathrm{Lag}(L)$ with respect to the C^1-topology.*

Under a restrictive condition, we also show the following:

Theorem 1.3. *In addition, suppose that $\mathrm{H}^1(P;\mathbb{R}) \to \mathrm{H}^1(L;\mathbb{R})$ is surjective. Then $\mathrm{Lag}(L)/\mathrm{Ham}(P,\Omega)$ is Hausdorff.*

The contents of this note are as follows. In section 2, we review some facts related to the flux conjecture and formulate our question. In section 3, we collect necessary results in [4]. We will prove our main theorem in section 4.

2. Preliminaries

From now on, we assume that Lagrangian submanifolds are closed and embedded. We recall the flux homomorphism and summarize some facts on the flux conjecture. For a one-parameter family $H = \{h_t\}$ of smooth functions on M, we have the Hamiltonian isotopy $\{\phi_t^H\}$, which is obtained by integrating the one-parameter family $\{X_{h_t}\}$ of Hamiltonian vector fields, i.e., $i(X_{h_t})\omega = dh_t$. A diffeomorphism ϕ is called a Hamiltonian diffeomorphism, if ϕ is expressed as the time-one map ϕ_1^H for some $H = \{h_t\}$. Denote by $\mathrm{Ham}(M,\omega)$ the group of all Hamiltonian diffeomorphisms of (M,ω), which is a significant subgroup of the group $\mathrm{Symp}(M,\omega)$ of all symplectomorphisms of (M,ω). Clearly, $\mathrm{Ham}(M,\omega)$ is contained in the identity component $\mathrm{Symp}_0(M,\omega)$.

We introduce some notation. Let θ be a closed 1-form on M. Denote by X_θ the symplectic vector field such that $i(X_\theta)\omega = \theta$. Let $\Theta = \{\theta_t\}$ be a one-parameter family of closed 1-forms. Denote by $\{\psi_t^\Theta\}$ the symplectic isotopy generated by $\{X_{\theta_t}\}$. Any symplectic isotopy $\{\psi_t\}$ such that $\psi_0 = id$ is described in this way. When Θ is not specified, we denote by $\{X_t\}$ the vector field generating $\{\psi_t\}$.

Let $\widetilde{\mathrm{Symp}}_0(M,\omega)$ be the universal covering group of $\mathrm{Symp}_0(M,\omega)$. Set

$$\widetilde{F}(\{\psi_t\}) = \int_0^1 [i(X_t)\omega]dt \in \mathrm{H}^1(M;\mathbb{R}).$$

Then it descends to a homomorphism

$$\widetilde{F} : \widetilde{\mathrm{Symp}_0}(M,\omega) \to \mathrm{H}^1(M;\mathbb{R}).$$

Denote by Γ_ω, which is called the flux group, the image of the kernel of the covering projection $\widetilde{\mathrm{Symp}_0}(M,\omega) \to \mathrm{Symp}_0(M,\omega)$, which is isomorphic to $\pi_1(\mathrm{Symp}_0(M,\omega))$, under $\widetilde{F}$. Then $\widetilde{F}$ induces the flux homomorphism

$$F : \mathrm{Symp}_0(M,\omega) \to \mathrm{H}^1(M;\mathbb{R})/\Gamma_\omega,$$

which is continuous with respect to the C^1-topology on the domain. It is known that $\mathrm{Ham}(M,\omega)$ coincides with the kernel of the flux homomorphism F. The flux conjecture stated in the introduction is equivalent to that Γ_ω is discrete in $\mathrm{H}^1(M;\mathbb{R})$ in the usual topology. In fact, this is the statement, which we proved in [5]. Hence, we have

$$\mathrm{Symp}_0(M,\omega)/\mathrm{Ham}(M,\omega) \cong \mathrm{H}^1(M;\mathbb{R})/\Gamma_\omega$$

is a Hausdorff space.

Consider the graph Gr_ψ of a symplectomorphism ψ as a Lagrangian submanifold in $(M \times M, -\mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega)$. The diagonal $\Delta \subset M \times M$ is the graph of the identity. When $\psi \in \mathrm{Symp}_0(M,\omega)$, Gr_ψ is Lagrangian isotopic to Δ, i.e., $Gr_\psi \in \mathrm{Lag}(\Delta)$. Thus we can identify the space $\mathrm{Symp}_0(M,\omega)/\mathrm{Ham}(M,\omega)$ with $\mathrm{Lag}^{\pitchfork}(\Delta)/(\{id\} \times \mathrm{Ham}(M,\omega))$, where $\mathrm{Lag}^{\pitchfork}(\Delta)$ is the subspace of $\mathrm{Lag}(\Delta)$ consisting of L transverse to all fibers of the first and second factor projections. We may ask whether $\mathrm{Lag}(\Delta)/\mathrm{Ham}(M \times M, -\mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega)$ is Hausdorff. This is a special case of Question 1.1. We come back to this question in the end of section 4.

Before proceeding further, we explain the C^1-topology on $\mathrm{Lag}(L)$ for the sake of reader's convenience. For $L' \in \mathrm{Lag}(L)$, we pick a tubular neighborhood U of L', which is certainly diffeomorphic to L, such that U is symplectomorphic to a neighborhood V of the zero section $O_{L'}$ in T^*L' (Weinstein neighborhood). Let $f : U \to V$ be a symplectomorphism such that $f(L') = O_{L'}$. Denote by S_η the cross section of T^*L' corresponding to a 1-form on L'. Pick and fix a Riemannian metric on L'. Note that the C^1-norm of differential forms depends on the choice of Riemannian metrics and is not preserved by self-diffeomorphisms. However, the C^1-topology is certainly well-defined, when the manifold is compact. For a sufficiently small $\epsilon > 0$, we set

$$\mathcal{U}_\epsilon(L') = \{f^{-1}(S_\eta) | \eta \text{ is a closed 1-form on } L', \ \| \eta \|_{C^1} < \epsilon\}.$$

The family of such sets gives a basis of the C^1-topology on $\mathrm{Lag}(L)$. As a warm-up to Question 1.1, we discuss the case that the ambient symplectic manifold is exact, e.g., the total space of the cotangent bundle equipped with the standard symplectic form. Firstly, we have the following:

Proposition 2.1. *Let $(X, d\lambda)$ be an exact symplectic manifold and $i : L \to X$ a Lagrangian embedding. For any $\phi \in \mathrm{Ham}(X, d\lambda)$, we have*

$$[i^*\lambda] = [(\phi \circ i)^*\lambda] \in \mathrm{H}^1(L;\mathbb{R}).$$

In particular, if $i : L \to X$ is an exact Lagrangian embedding, i.e., $i^*\lambda$ is an exact 1-form, then $\phi \circ i$ is also an exact Lagrangian embedding.

Proof. Let $\{\phi_t^H\}$ be a Hamiltonian isotopy generated by $H = \{h_t\}$. Then we have

$$\begin{aligned} \frac{d}{dt}(\phi_t^H \circ i)^*\lambda &= i^* \circ \frac{d}{dt}\phi_t^{H*}\lambda \\ &= i^* \circ \phi_t^{H*} L_{X_{h_t}}\lambda \\ &= d\left(i^* \circ \phi_t^{H*}(i(X_{h_t})\lambda + h_t)\right). \end{aligned}$$

Thus $i^*\lambda$ and $(\phi \circ i)^*\lambda$ are cohomologous.

If $i^*\lambda$ is an exact 1-form, $(\phi_t^H \circ i)^*\lambda$ is also exact. Namely, $\phi_1^H \circ i$ is an exact Lagrangian embedding. □

Remark 2.1. If a symplectic manifold (P, Ω) satisfies $[\Omega] \in \mathrm{Im}\left(\mathrm{H}^2(P;\mathbb{Z}) \to \mathrm{H}^2(P;\mathbb{R})\right)$, there exists a so-called prequantum line bundle $L \to P$. A Lagrangian submanifold $L \subset P$ is called exact (a Bohr-Sommerfeld orbit), if L can be lifted to a Legendrian submanifold in the unit circle bundle $S(L)$. The above argument is easily extended to show that Hamiltonian deformations of exact Lagrangian submanifolds are exact.

Let $i_t : L \to X$ be a one-parameter family of exact Lagrangian embeddings. Then we can find a Hamiltonian isotopy $\{\phi_t^H\}$ of the ambient exact symplectic manifold such that $i_t = \phi_t^H \circ i$. In general, a one-parameter family of Lagrangian embeddings may not extend to a symplectic isotopy of the ambient symplectic manifold. We have the following:

Lemma 2.2. *Let (P, Ω) be a symplectic manifold and L an embedded closed connected Lagrangian submanifold. If the embedding $i : L \to P$ induces a surjective homomorphism $i^* : \mathrm{H}^1(P;\mathbb{R}) \to \mathrm{H}^1(L;\mathbb{R})$, then any one-parameter deformation of Lagrangian embedding i extends to an ambient symplectic isotopy of (P, Ω) starting from id. In other words, the action of $\mathrm{Symp}_0(P, \Omega)$ on $\mathrm{Lag}(L)$ is transitive.*

In the case of the zero section O_X in the cotangent bundle T^*X equipped with the standard symplectic form $d\lambda$, we have the following:

Proposition 2.3. *There is a canonical continuous bijection*

$$\mathrm{Lag}(O_X)/\mathrm{Ham}(T^*X, d\lambda) \cong \mathrm{H}^1(L;\mathbb{R}),$$

hence $\mathrm{Lag}(O_X)/\mathrm{Ham}(T^*X, d\lambda)$ is Hausdorff.

Proof. Firstly, we define $G : \mathrm{Lag}(O_X) \to \mathrm{H}^1(L;\mathbb{R})$ as follows. Let $L \in \mathrm{Lag}(O_X)$. For a loop γ in O_X, we pick a loop γ' in L such that γ and γ' are homologous in T^*X. Pick a 2-chain C such that $\partial C = \gamma' - \gamma$. Then we set

$$\begin{aligned} \langle G(L), [\gamma] \rangle &= \int_C d\lambda \\ &= \int_{\gamma'} \lambda - \int_\gamma \lambda. \end{aligned}$$

Let θ be a closed 1-form and $\{\psi_1^\theta\}$ the corresponding symplectic isotopy, i.e., the addition of $t\theta$ fiberwisely. Then $G\left(\psi_1^\theta(O_X)\right) = [\theta]$, so G is surjective.

Let $L', L'' \in \mathrm{Lag}(O_X)$ such that $G(L') = G(L'')$. Choose $\Theta' = \{\theta'_t\}, \Theta'' = \{\theta''_t\}$ such that $L' = \psi^{\Theta'}(O_X)$ and $L'' = \psi^{\Theta''}(O_X)$. Note that

$$G(L') = \int_0^1 [\theta'_t] dt \quad \text{and} \quad G(L'') = \int_0^1 [\theta''_t] dt,$$

which are equal to each other. Consider $\Theta(s) = \{(1-s)\theta'_t + s\theta''_t\}$. Then the time-one maps $\psi_1^{\Theta(s)}$ are Hamiltonian isotopic. Therefore L' and L'' are Hamiltonian isotopic. Hence G induces a bijection $\overline{G} : \mathrm{Lag}(O_X) \to \mathrm{H}^1(L;\mathbb{R})$. Since G is continuous, $\mathrm{Lag}(O_X)/\mathrm{Ham}(T^*X, d\lambda)$ is Hausdorff. □

3. Floer theory for Lagrangian submanifolds

We summarize some results from [4], which are necessary for our later argument. In this section, a Lagrangian submanifold L, resp. a pair $(L^{(0)}, L^{(1)})$ of closed embedded Lagrangian submanifolds, is equipped with a relative spin structure (see §44 in [4]). For such an L, we constructed a filtered A_∞-algebra associated to L, which satisfies the gapped condition. Although it depends on various auxiliary choices, its homotopy type is uniquely determined (Theorems 10.11 and 14.1 in [4]). By the canonical model theorem (Theorem 23.2 in [4]), we can reduce the filtered A_∞-structure to the $\overline{\mathfrak{m}}_1$-cohomology, namely we have the following:

Theorem 3.1 (see Theorem A in [4])**.** *To each relatively spin Lagrangian submanifold L we can associate a structure of filtered A_∞-algebra $\{\mathfrak{m}_k\}$ on $\mathrm{H}^*(L;\Lambda_{0,nov})$, which is well-defined up to isomorphism. A symplectomorphism $\psi : (P, L) \to (P', L')$ induces an isomorphism of filtered A_∞-algebras, whose homotopy class depends only on ψ.*

Here $\Lambda_{0,nov}$ is a graded ring called the universal Novikov ring (see Definition 6.2 in [4]). We introduce two generators e and T such that $\deg e = 0$ and $\deg T = 0$.

$$\begin{aligned}
\Lambda_{nov} &= \{\sum_i a_i e^{\mu_i} T^{\lambda_i} \mid a_i \in \mathbb{Q},\ \mu_i \in \mathbb{Z},\ \lambda_i \in \mathbb{R},\ \lambda_i \to +\infty (i \to +\infty)\} \\
\Lambda_{0,nov} &= \{\sum_i a_i e^{\mu_i} T^{\lambda_i} \in \Lambda_{nov} \mid \lambda_i \geq 0\} \\
\Lambda_{0,nov}^+ &= \{\sum_i a_i e^{\mu_i} T^{\lambda_i} \in \Lambda_{nov} \mid \lambda_i > 0\}
\end{aligned}$$

We will also use the subring consisting of elements of degree zero:

$$\Lambda_{nov}^{\deg=0} = \{\sum_i a_i T^{\lambda_i} \mid a_i \in \mathbb{Q},\ \lambda_i \in \mathbb{R},\ \lambda_i \to +\infty (i \to +\infty)\}.$$

Let $(L^{(0)}, L^{(1)})$ be a relative spin pair of Lagrangian submanifolds, which is of clean intersection. We constructed a filtered A_∞-bimodule over the filtered A_∞-algebras associated to $L^{(1)}$ and $L^{(0)}$ (see Theorem F in [4]). However, in general, the Floer complex of the pair $(L^{(0)}, L^{(1)})$ may not be constructed, since the bubbling-off of pseudo-holomorphic discs can happen as codimension one phenomena. We developed obstruction theory in order to define the Floer complex for a pair of Lagrangian submanifolds.

Theorem 3.2 (see Theorems B and G in [4])**.** *To each relatively spin Lagrangian submanifold $L \subset P$, we can associate a set $\mathcal{M}_{\mathrm{weak,def}}(L)$, which may be empty, and maps*

$$\begin{aligned} \pi_{\mathrm{amb}} &: \mathcal{M}_{\mathrm{weak,def}}(L) \to \mathrm{H}^2(P; \Lambda_{0,nov}) \\ \mathfrak{PO} &: \mathcal{M}_{\mathrm{weak,def}}(L) \to \Lambda^+_{0,nov} \end{aligned}$$

with the following properties.
(1) A symplectomorphism $\psi : (P, L) \to (P', L')$ induces a map

$$\psi_* : \mathcal{M}_{\mathrm{weak,def}}(L) \to \mathcal{M}_{\mathrm{weak,def}}(L')$$

such that $\pi_{\mathrm{amb}} \circ \psi_* = (\psi^{-1})^* \circ \pi_{\mathrm{amb}}$ and $\mathfrak{PO} \circ \psi_* = \mathfrak{PO}$.
(2) For $(\mathbf{b}_1, \mathbf{b}_0) \in \mathcal{M}_{\mathrm{weak,def}}(L^{(1)}) \times_{\pi_{\mathrm{amb}}, \mathfrak{PO}} \mathcal{M}_{\mathrm{weak,def}}(L^{(0)})$, we can associate the Floer cohomology $\mathrm{HF}^*\big((L^{(1)}, \mathbf{b}_1), (L^{(0)}, \mathbf{b}_0); \Lambda_{0,nov}\big)$. After extending the coefficient ring from $\Lambda_{0,nov}$ to Λ_{nov}, we have $\mathrm{HF}^*\big((L^{(1)}, \mathbf{b}_1), (L^{(0)}, \mathbf{b}_0); \Lambda_{nov}\big)$.
(3) Let $\{\phi^H_t\}$ be a Hamiltonian isotopy generated by H and

$$(\mathbf{b}_1, \mathbf{b}_0) \in \mathcal{M}_{\mathrm{weak,def}}(L^{(1)}) \times_{\pi_{\mathrm{amb}}, \mathfrak{PO}} \mathcal{M}_{\mathrm{weak,def}}(L^{(0)}).$$

Then $\{\phi^H_t\}$ induces an isomorphism

$$(\{\phi^H_t\}, id)_* : \mathrm{HF}^*\Big((L^{(1)}, \mathbf{b}_1), (L^{(0)}, \mathbf{b}_0); \Lambda_{nov}\Big) \to \mathrm{HF}^*\Big((\phi^H_1(L^{(1)}), \phi^H_{1*}\mathbf{b}_1), (L^{(0)}, \mathbf{b}_0); \Lambda_{nov}\Big).$$

Definition 3.1. A relatively spin Lagrangian submanifold L is said to be unobstructed after infinitesimal deformation, or bulk/boundary deformation, if $\mathcal{M}_{\mathrm{weak,def}}(L) \neq \emptyset$. The element $\mathbf{b} \in \mathcal{M}_{\mathrm{weak,def}}(L)$ is called a solution of the Maurer-Cartan equation in the filtered A_∞-algebra associated to L. If there is $\mathbf{b} \in \mathcal{M}_{\mathrm{weak,def}}(L)$ such that $\pi_{\mathrm{amb}}(\mathbf{b}) = 0$ and $\mathfrak{PO}(\mathbf{b}) = 0$, then we simply call L unobstructed.

Remark 3.1. In the case that the Maslov class vanishes on $\pi_2(P, L)$, the potential function $\mathfrak{PO}$ is automatically zero.

We have not yet found satisfactory sufficient condition for unobstructedness (after infinitesimal deformation), but have the following:

Theorem 3.3 (Theorem C in [4])**.** *There exist a series of positive integers m_k and classes*

$$[o^{2m_k}_k(L; \mathrm{weak}, \mathrm{def})] \in \frac{\mathrm{H}^{2m_k}(L; \mathbb{Q})}{\mathrm{Im}\,(i^*\mathrm{H}^{2m_k}(P; \mathbb{Q}) \to \mathrm{H}^{2m_k}(L; \mathbb{Q}))},$$

$k = 1, 2, \ldots$, (obstruction classes) such that if $[o^{2m_k}_k(L; \mathrm{weak}, \mathrm{def})]$ are all zero, then $\mathcal{M}_{\mathrm{weak,def}}(L)$ is non-empty. The number $2 - 2m_k$ is a sum of the Maslov indices of a finite collection of homotopy classes in $\pi_2(P, L)$ realized by pseudo-holomorphic discs with respect to a compatible almost complex structure on P.

In particular, we find the following:

Corollary 3.4. *If $i^* : \mathrm{H}^{2m_k}(P;\mathbb{Q}) \to \mathrm{H}^{2m_k}(L;\mathbb{Q})$ is surjective, $\mathcal{M}_{\mathrm{weak,def}}(L) \neq \emptyset$.*

Corollary 3.5. *If the Maslov class $\mu_L : \pi_2(P, L) \to \mathbb{Z}$ is zero, all obstruction classes $[o_k^{2m_k}(L; \mathrm{weak}, \mathrm{def})]$ lie in $\mathrm{H}^2(L;\mathbb{Q})$.*

We have the following non-vanishing result.

Theorem 3.6 (see Theorem E, Theorem 24.12 in [4])**.** *Suppose that $\mathbf{b} \in \mathcal{M}_{\mathrm{weak,def}}(L)$ and the Maslov indices of all pseudo-holomorphic discs bounding L are non-positive. Then the Poincare dual $PD[pt]$ of the point class gives non-trivial element in the Bott-Morse Floer cohomology $\mathrm{HF}^*((L,\mathbf{b})(L,\mathbf{b});\Lambda_{nov})$. The Poincare dual $PD[L]$ of the fundamental class of L can be deformed to a non-trivial element $PD[L]'$ of $\mathrm{HF}^*((L,\mathbf{b}),(L,\mathbf{b});\Lambda_{nov})$ such that $PD[L]' \equiv PD[L]$ mod $\Lambda^+_{0,nov}$.*

If the Maslov class μ_L vanishes on $\pi_2(P, L)$, we can equip L with a grading structure (a graded Lagrangian submanifold) in the sense of Kontsevich and Seidel [7]. We denote a graded Lagrangian submanifold by (L, σ). Changing the grading structure shifts the degree by a constant. For a relatively spin pair $(L^{(0)}, L^{(1)})$ of graded Lagrangian submanifolds with $\mathbf{b}_i \in \mathcal{M}_{\mathrm{weak,def}}(L^{(i)})$, we can construct the Floer cohomology $\mathrm{HF}^*\left((L^{(1)},\mathbf{b}_1,\sigma_1),(L^{(0)},\mathbf{b}_0,\sigma_0);\Lambda_{nov}^{\deg=0}\right)$ over $\Lambda_{nov}^{\deg=0}$. We also have the Floer cohomology over $\Lambda_{0,nov}^{\deg=0} = \Lambda_{0,nov} \cap \Lambda_{nov}^{\deg=0}$. A Lagrangian isotopy $\{L_t\}$ induces the one-to-one correspondence between grading structures on $L = L_0$ to those on $L' = L_1$. All results above hold for these Floer cohomologies for graded Lagrangian submanifolds. Then the degree of $PD[pt]$, resp. $PD[L]'$, is n, resp. 0. Here $n = \dim L$.

4. Proof of Theorem 1.1

Let $L \subset P$ be a relatively spin Lagrangian submanifold and $L_* \in \mathrm{Lag}(L)$. It is obvious that L_* is diffeomorphic to L. Pick a tubular neighborhood U of L_*, which is symplectomorphic to a tubular neighborhood V of the zero section O_{L_*} of T^*L_*, i.e., there exists a symplectomorphism $f : (U, L_*) \to (V, O_{L_*})$. Then a Lagrangian submanifold L', which is sufficiently close to L_* in the C^1-topology, is described by a closed 1-form on L_*. If two Lagrangian submanifolds L_1, L_2 are sufficiently close to L_* in the C^1-topology and correspond to cohomologous closed 1-forms, L_2 can be realized by a Hamiltonian deformation of L_1. Hence small Lagrangian deformations of L' up to Hamiltonian deformations are parametrized by a small neighborhood $W \subset \mathrm{H}^1(L;\mathbb{R})$ of the origin. We announced the following statement in the case that $L_* = L$ in [6].

Proposition 4.1. *Let $L \subset P$ be a relatively spin Lagrangian submanifold such that the Maslov class μ_L vanishes on $\pi_2(P, L)$ and $L_* \in \mathrm{Lag}(L)$. Suppose that L is unobstructed after infinitesimal deformation, i.e., there exists $\mathbf{b} \in \mathcal{M}_{\mathrm{weak,def}}(L)$. Let $\{\phi_t^H\}$ be a Hamiltonian isotopy generated by H. Suppose that L and $\phi_1^H(L)$ are sufficiently close to L_* in the C^1-topology, hence $f(L)$ and $f \circ \phi_1^H(L)$ are the graphs of closed 1-forms η and η' on L. Then η' is cohomologous to η.*

Before giving the proof, we prepare some notation. Let θ be a closed 1-form on L_*. By abuse of notation, we denote by θ the closed 1-form $f^* \circ \mathrm{pr}^*(\theta)$ on the tubular neighborhood U of L_*, where pr is the projection of the cotangent bundle of L_*. Then denote by X^θ the vector field on U such that $i(X^\theta)\Omega = \theta$ and by $\{\psi_t^\theta\}$ the symplectic local flow generated by X^θ. Once a Riemannian metric is chosen on L_*, we have a one-to-one correspondence between vector fields and 1-forms. We denote by $\theta^\#$ the vector field such that

$$\langle \theta^\#, \cdot \rangle = \theta(\cdot).$$

Proof. Firstly, we note that $L' = \phi_1^H(L)$ is also unobstructed after infinitesimal deformation, i.e., $\mathbf{b}' = \phi_{1*}^H(\mathbf{b}) \in \mathcal{M}_{\mathrm{weak,def}}(L')$. We also have $\pi_{amb}(\mathbf{b}) = \pi_{amb}(\mathbf{b}')$ and $\mathfrak{PO}(\mathbf{b}) = \mathfrak{PO}(\mathbf{b}')$. (Under the assumption that the Maslov class vanishes on $\pi_2(P, L)$, the latter is zero.)

Since μ_L vanishes on $\pi_2(P, L)$, there is a grading structure σ on L. A Lagrangian deformation $\{L_t\}$ such that $L' = L_1$ induces the grading structure $\sigma^{\{L_t\}}$ on L'. In particular, the Hamiltonian isotopy $\{\phi_t^H\}$ induces the grading structure $\sigma^H = \sigma^{\{\phi_t^H(L)\}}$.

If L and L' are sufficiently C^1-close to L_*, we can regard L, resp. L', as the graph of a closed 1-form η, resp. η' in T^*L_*. Clearly $L' = \psi_1^{\eta'-\eta}(L)$. The Lagrangian deformation $\{\psi_t^{\eta'-\eta}(L)\}$ induces the grading structure $\sigma^{\eta,\eta'} = \sigma^{\{\psi_t^{\eta'-\eta}(L)\}}$ on L'. Two grading structures σ^H and $\sigma^{\eta,\eta'}$ may not coincide, hence the degrees in Floer theory may be shifted by a constant. Namely, there is an integer c such that $\mathrm{HF}^*\left((L', \mathbf{b}', \sigma^H), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0}\right)$ is isomorphic to $\mathrm{HF}^{*+c}\left((L', \mathbf{b}', \sigma^{\eta,\eta'}), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0}\right)$ as graded $\Lambda_{nov}^{\deg=0}$-modules.

Now we explain, more precisely, the condition how close L and L' are to L_* in the C^1-topology. Namely, $\| \eta \|_{C^1}, \| \eta' \|_{C^1} < \epsilon$ for some $\epsilon > 0$, which we will specify later. Pick and fix a Riemannian metric on L_*. We may assume that $V \subset T^*L_*$ is the δ-neighborhood of the zero section O_{L_*} with respect to the induced metric. Denote by V' the $2\delta/3$-neighborhood of O_{L_*} and set $U' = f^{-1}(V')$. Pick an almost complex structure J compatible with Ω. By the monotonicity lemma for J-holomorphic curves, there exists a constant $e > 0$ such that $\int_S \Omega > e$ for any non-constant compact J-holomorphic curve in $P \setminus U'$ with $\partial S \subset \partial U'$. We will choose $\delta > 0$ sufficiently small later. Note that the constant $e > 0$ does not depend on the choice of δ, whenver $\partial U'$ is of contact type. (By the maximal principle, such a J-holomorphic curve is not contained in any Weinstein neighborhood of L_*.)

Choose a Morse function h on L_* such that $|dh| < \delta/3$. We may assume, without loss of generality, that h has a unique maximum and a unique minimum. We also assume that $\max h - \min h < e/4$. Next we pick contractible neighborhoods $W' \Subset W$ of the set of critial points of h and a smooth cut off function ρ such that the diameter of W is bounded by $1/\ell$ and

$$\rho = 1 \text{ on } W', \quad \rho = 0 \text{ on } L_* \setminus W, \quad |d\rho| < 2\ell,$$

where K is a positive number.

The first condition for $\epsilon > 0$ is

$$\epsilon < \frac{1}{100} \min_{L_* \setminus W'} |dh|. \tag{1}$$

In particular, this condition implies that $\epsilon < \delta/300$.

Suppose that the closed 1-forms η and η' satisfy that $\| \eta \|_{C^0}, \| \eta' \|_{C^0} < \epsilon$. Since W is contractible, we find that $(\eta' - \eta)|_W = dg$ for some smooth function g on W. We may assume that $|g| < 2\epsilon/\ell$, since $|dg| < 2\epsilon$ and the diameter of W is at most $1/\ell$. We regard $\rho \cdot g$ as a smooth function on L_*. Then $\eta'' = dh + \eta' - d(\rho \cdot g)$ is also a closed 1-form cohomologous to η'.

It is easy to see that

$$|\eta''| \leq |dh| + |\eta'| + |d(\rho \cdot g)| < |dh| + 7\epsilon,$$

which is smaller than $2\delta/3$. Thus $S_{\eta''}$ is contained in V'. It guarantees a Hamiltonian isotopy $\{\phi_t^K\}$, which maps $f^{-1}(S_{\eta'})$ to $f^{-1}(S_{\eta''})$, such that the support of K is contained in $U' \cong V'$. Note also that

$$|dh - (\eta'' - \eta)| < 6\epsilon.$$

By the construction, we find that S_η and $S_{\eta''}$ intersect transversally and their intersection points are identified with the critical points of h.

Since (1) holds on $L_* \setminus W'$ and $\eta'' - \eta = dh$ on W',

$$|dh - (\eta'' - \eta)| < \frac{1}{10}|dh|, \tag{2}$$

we find that

$$dh\left((\eta'' - \eta)^\#\right) \geq \frac{9}{10} dh\left((dh)^\#\right).$$

Hence the function h increases along integral curves of $(\eta'' - \eta)^\#$, i.e., h is a Liapunov function.

Now we give the second condition for ϵ through δ. Note that $\| \eta \|_{C^1}, \| \eta'' \|_{C^1} < 2\delta/3$. Pick $\delta > 0$ sufficiently small, there is a diffeomorphism Ψ supported in V' such that $\Psi(O_{L_*}) = S_\eta$ and $\Psi(S_{dh}) = S_{\eta''}$ with the following property.

- Let $\{J_t^h\}$ be the t-dependent family of almost complex structures on $V \subset T^*L_*$ associated to h, which Floer introduced in [2]. Ψ is C^1-close enough to the identity such that the family $\widehat{J} = \{\Psi_* J_t^h\}$ of almost complex structures is tamed by the canonical symplectic structure on T^*L_*.

We arrange the almost complex structure J on P, which may depend on $t \in [0,1]$, so that J coincides with $\widehat{J}$ on U'. Then all Floer connecting trajectories, i.e., J-holomorphic strip u bounded by $L = f^{-1}(S_\eta)$ and $L'' = f^{-1}(S_{\eta''})$, contained in U correspond to the bounded solution of

$$\dot{\gamma}(\tau) = (\eta'' - \eta)^\# (\gamma(\tau)).$$

Then the energy $E(u)$ of the Floer connecting trajectory u satisfies

$$\begin{aligned} E(u) &= \int_{\mathbb{R}\times[0,1]} u^*\omega_{\mathrm{can}} \\ &= \int_{-\infty}^{+\infty} \gamma^*(\eta''-\eta), \end{aligned}$$

where ω_{can} is the standard symplectic form on T^*L_*. By (2), we have

$$(\eta''-\eta)\left((\eta''-\eta)^{\#}\right) = |(\eta''-\eta)^{\#}|^2 < \frac{9}{8}dh\left((\eta''-\eta)^{\#}\right).$$

Thus we find that

$$E(u) < \frac{9}{8}\left(h(p^+)-h(p^-)\right) < \frac{e}{2}.$$

Here $p^{\pm} = \lim_{\tau\to\pm\infty}\gamma(\tau)$, which are critical points of h. Thus all Floer connecting trajectories for (L, L'') contained in U have symplectic area less than $e/2$. Note also that ∂U is a boundary of contact type.

We fix $\epsilon > 0$ satisfying the above conditions. Note that, if a Floer connecting trajectory is not contained in U, its symplectic area must be more than e.

Denote by $H\#K$ the time-dependent Hamiltonian, which generates the concatenation of $\{\phi_t^H\}$ and $\{\phi_t^K\}$. Write $(L'', \mathbf{b}'', \sigma'') = (\phi_1^{H\#K}(L), \phi_{1*}^{H\#K}\mathbf{b}, \phi_{1*}^K\sigma^{\eta,\eta'})$. Now we consider the Floer complex $\mathrm{CF}^*\left((L'',\mathbf{b}'',\sigma''),(L,\mathbf{b},\sigma);\Lambda_{nov}^{\deg=0}\right)$. The Floer coboundary operator $\delta^{\mathbf{b}'',\mathbf{b}}$ decomposes into

$$\delta^{\mathbf{b}'',\mathbf{b}} = \delta^{\mathbf{b}'',\mathbf{b}}_{<e} + \delta^{\mathbf{b}'',\mathbf{b}}_{\geq e}.$$

Here $\delta^{\mathbf{b}'',\mathbf{b}}_{<e}$ counts Floer connecting trajectories of relative index 1 and with energy less than e, i.e, those contained in U, while $\delta^{\mathbf{b}'',\mathbf{b}}_{\geq e}$ is other contributions to $\delta^{\mathbf{b}'',\mathbf{b}}$.

The operator $\delta^{\mathbf{b}'',\mathbf{b}}_{<e}$ can be identified with the coboundary operator of the Novikov complex for the closed 1-form $\eta''-\eta$.

Denote by p_{min} and p_{max} the unique minimum and maximum of h, hence, the unique zero of $\eta''-\eta$ of index 0, n, respectively. If $\eta''-\eta$, hence $\eta'-\eta$, is not exact, the Novikov cohomology $\mathrm{HN}^*(\eta''-\eta)$ vanishes in degrees $* = 0, n$. Let $\overline{p}_{min}, \overline{p}_{max}$ be lifts of p_{min}, p_{max} to the abelian covering space of L_* associated to the de Rham cohomology class $[\eta''-\eta] = [\eta'-\eta]$. Hence $\delta^{\mathbf{b}'',\mathbf{b}}_{<e}\overline{p}_{min} \neq 0$, and there are $\overline{q}_i$ of index n such that $\delta^{\mathbf{b}'',\mathbf{b}}_{<e}(\sum_i a_i\overline{q}_i) = \overline{p}_{max}$.

Since p_{min} is the unique zero of $\eta''-\eta$ with index 0 and the Floer coboundary operator $\delta^{\mathbf{b}'',\mathbf{b}}$ is linear over the field $\Lambda_{nov}^{\deg=0}$, there does not exist a non-trivial Floer cocycle of degree 0.

Note also that p_{max} is the unique zero of $\eta''-\eta$ with index n. Using the energy filtration, we can find that any Floer cocycle of degree n is a Floer coboundary. Therefore we find that

$$\mathrm{HF}^*\left((L'',\mathbf{b}'',\sigma''),(L,\mathbf{b},\sigma);\Lambda_{nov}^{\deg=0}\right) = 0$$

for $* = 0, n$. Because the Maslov class μ_L vanishes on $\pi_2(P, L)$, the degree of $L \cap L''$ is determined by the index of zeros of $\eta'' - \eta$. Hence the degrees are between 0 and n. Combining these, we find that if

$$\mathrm{HF}^* \left((L'', \mathbf{b}'', \sigma''), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0} \right) \neq 0$$

for $* = k, \ell$, then $|k - \ell| < n - 1$.

By the hypothesis that μ_L vanishes on $\pi_2(P, L)$, the Maslov indices of all pseudo-holomorphic discs are zero. Thus Theorem 3.6 guarantees existence of non-trivial classes $PD[pt]$ and $PD[L]'$ in $\mathrm{HF}^* \left((L, \mathbf{b}, \sigma), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0} \right)$ with $* = n, 0$, respectively.

By the graded Lagrangian version of Theorem 3.2 (3), we find that

$$\mathrm{HF}^* \left((L'', \mathbf{b}'', \sigma^{H\#K}), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0} \right) \cong \mathrm{HF}^* \left((L, \mathbf{b}, \sigma), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0} \right).$$

Therefore there are k and ℓ such that

$$\mathrm{HF}^* \left((L'', \mathbf{b}'', \sigma^{H\#K}), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0} \right) \neq 0$$

for $* = k, \ell$ and $k - \ell = n$.

Although the grading structures $\sigma^{H\#K}$ and σ'' may be different, the Floer connecting trajectories do not depend on the choice of a grading structure and the resulting cohomology groups as graded modules are isomorphic up to a degree shift. Namely, there is an integer c such that

$$\mathrm{HF}^* \left((L'', \mathbf{b}'', \sigma^{\ H\#K}), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0} \right) \cong \mathrm{HF}^{*+c} \left((L'', \mathbf{b}'', \sigma''), (L, \mathbf{b}, \sigma); \Lambda_{nov}^{\deg=0} \right).$$

Hence we obtain a contradiction. □

Now we are ready to finish

Proof of Theorem 1.1. Recall that $f : U \to V$ is a Weinstein neighborhood of L_*. Set

$$\mathcal{U}_\epsilon = \{ f^{-1}(S_\theta) \mid \theta \text{ is a closed 1-form on } L_*, \quad \| \theta \|_{C^1} < \epsilon \},$$

which is a neighborhood of L_* in $\mathrm{Lag}(L)$. Then Proposition 4.1 implies that

$$\mathrm{Ham}(P, \Omega) \cdot L \cap \mathcal{U}_\epsilon$$

consists of $f^{-1}(S_\theta)$ such that $\| \eta \|_{C^1} < \epsilon$ and θ belongs to a unique cohomology class $a \in \mathrm{H}^1(L_*; \mathbb{R})$.

If L and L_* belong to different $\mathrm{Ham}(P, \Omega)$-orbits, then the class a is non-zero. Then there exists a constant $d > 0$ such that $\| \theta \|_{C^0} > d$, hence $\| \theta \|_{C^1} > d$, for any θ representing the class a. If $\epsilon < d$, then

$$\mathrm{Ham}(P, L) \cdot L \cap \mathcal{U}_\epsilon = \emptyset.$$

Therefore $\mathrm{Ham}(P, L) \cdot L$ is closed in $\mathrm{Lag}(L)$ with respect to the C^1-topology. □

Remark 4.1. The above proof is similar to the proof of Theorem 4.3 in [9].

By Corollary 3.5, if a relatively spin Lagrangian submanifold L satisfies that

$$\mathrm{H}^2(L;\mathbb{Q}) = 0$$

and the Maslov class vanishes on $\pi_2(P, L)$, L is unobstructed. Hence Corollary 1.2 follows from Theorem 1.1.

Restrict ourselves to unobstructed Lagrangian submanifolds and denote by $\mathrm{Lag}^{\mathrm{unob}}(L)$ the subspace of unobstructed Lagrangian submanifolds in $\mathrm{Lag}(L)$. Note that this space may no longer connected. It may be more natural to ask the following:

Question 4.1. Is $\mathrm{Lag}^{\mathrm{unob}}(L)/\mathrm{Ham}(P,\Omega)$ Hausdorff?

Proof of Theorem 1.3. Let $L_1, L_2 \in \mathrm{Lag}(L)$ such that $\phi(L_1) \neq L_2$ for any $\phi \in \mathrm{Ham}(P,\Omega)$. We will find neighborhoods $\mathcal{U}_i$ of L_i in the C^1-topology such that $\mathrm{Ham}(P,\Omega)\cdot\mathcal{U}_i$, $i = 1, 2$, are disjoint, which is equivalent to that

$$\mathcal{U}_1 \cap \mathrm{Ham}(P,\Omega)\cdot\mathcal{U}_2 = \emptyset. \tag{3}$$

Let $f_i : U(L_i) \to V(O_{L_i})$ be Weinstein's standard form around the Lagrangian submanifold L_i. We assume that f_i, $i = 1, 2$, extend to slightly larger Weinstein neighborhoods. (This condition makes the smooth extension of θ_i to $\widetilde{\theta}_i$ exist in a later argument.) Pick and fix a Riemannian metric on L_i. We set

$$\mathcal{V}_{i,\epsilon} = \{S_\theta \mid \theta \text{ is a closed 1-form on } L_i,\ \| \theta \|_{C^1(\Omega^1(L_i))} < \epsilon\}.$$

Pick $\bar{\epsilon}_i > 0$ such that $V(O_{L_i})$ contains the $\bar{\epsilon}_i$-neighborhood of O_{L_i} with respect to the induced metric. By Theorem 1.1, there is small $\epsilon_1 > 0$ such that $2\epsilon_1 < \bar{\epsilon}_1$ and

$$\mathrm{Ham}(P,\Omega)\cdot L_2 \cap \widetilde{\mathcal{U}}_1 = \emptyset,$$

where

$$\widetilde{\mathcal{U}}_1 = \{f^{-1}(S) \mid S \in \mathcal{V}_{1,2\epsilon_1}\}.$$

Since L_1 and L_2 are the images of isotopic embeddings, we find that

$$\ker\left(\mathrm{H}^1(P;\mathbb{R}) \to \mathrm{H}^1(L_1;\mathbb{R})\right) = \ker\left(\mathrm{H}^1(P;\mathbb{R}) \to \mathrm{H}^1(L_2;\mathbb{R})\right).$$

We denote by $\mathcal{K}$ this submodule of $\mathrm{H}^1(P;\mathbb{R})$. Although there are no canonical choice of parametrizing L_i, $i = 1, 2$, we can identify $\mathrm{H}^1(L_1;\mathbb{R})$ and $\mathrm{H}^1(L_2;\mathbb{R})$ through $\mathrm{H}^1(P;\mathbb{R})/\mathcal{K}$.

Choose a neighborhood $W \subset \mathrm{H}^1(P;\mathbb{R})/\mathcal{K}$, which we identify with $\mathrm{H}^1(L_i,\mathbb{R})$, $i = 1, 2$, of the origin such that any cohomology class in W is represented by some θ on L_1 with $\| \theta \|_{C^1(\Omega^1(L_1))} < \epsilon_1$. Since the de Rham cohomology class is determined by the integration on cycles generating the first homology group, we can also find a sufficiently small $\epsilon_2 > 0$ such that $\epsilon_2 < \bar{\epsilon}_2$ and $[\theta] \in W$ for any closed 1-form θ on L_2 with $\| \theta \|_{C^1(\Omega^1(L_2))} < \epsilon_2$.

Write

$$\mathcal{U}_i = \{f^{-1}(S) \mid S \in \mathcal{V}_{i,\epsilon_i}\},$$

and prove (3). Suppose that there exist $L' \in \mathcal{U}_1$, $L'' \in \mathcal{U}_2$ and $\phi \in \mathrm{Ham}(P,\Omega)$ such that $L' = \phi(L'')$. Let θ_i be closed 1-form on L_i such that $f_1(L') = S_{\theta_1}$ and $f_2(L'') = S_{\theta_2}$, in particular $\| \theta_i \|_{C^1(\Omega^1(L_i))} < \epsilon_i$. We extend θ_i to the tubular neighborhoods $U(L_i)$

by $f_i^* \circ \pi_i^*(\theta_i)$, where π_i is the projection of the cotangent bundle of L_i. Using the hypothesis that $\mathrm{H}^1(P;\mathbb{R}) \to \mathrm{H}^1(L_i;\mathbb{R})$ is surjective, we extend them to closed 1-forms $\widetilde{\theta}_i$ on P. Note that $\psi_1^{\widetilde{\theta}_2}(L_2) = L''$. By the choice of ϵ_2, the cohomology class $[\theta_2]$ belongs to W. Since ϕ is isotopic to the identity, $[\phi^*(\widetilde{\theta}_2)] \mod \mathcal{K}$ belongs to W. Pick a closed 1-form θ_2' on L_1 in this class such that $\| \theta_2' \|_{C^1(\Omega^1(L_1))} < \epsilon_1$. Then extend it to $\widetilde{\theta}_2'$ on P such that $[\widetilde{\theta}_2'] = [\widetilde{\theta}_2] \in \mathrm{H}^1(P;\mathbb{R})$ and $\widetilde{\theta}_2'$ coincides with $f_1^* \circ \pi_1^*(\theta_2')$ on $U(L_1)$. Note that $\| \theta_1 - \theta_2' \|_{C^1(\Omega(L_1))} < 2\epsilon$. Since $\phi \in \mathrm{Ham}(P,\Omega)$ and $[\widetilde{\theta}_2'] = [\widetilde{\theta}_2]$, $f_1^{-1}(S_{\theta_1 - \theta_2'}) = \psi_1^{\widetilde{\theta}_1 - \widetilde{\theta}_2'}(L_1)$ is Hamiltonian isotopic to L_2. It implies that there is a Hamiltonian isotopy $\{\phi_t^H\}$ such that $\phi_1^H(L_2) \in \widetilde{\mathcal{U}}_1$, which is a contradiction. □

Under the hypothesis that $\mathrm{H}^*(P;\mathbb{R}) \to \mathrm{H}^*(L;\mathbb{R})$ is surjective, all $L' \in \mathrm{Lag}(L)$ are unobstructed, i.e., $\mathrm{Lag}(L) = \mathrm{Lag}^{\mathrm{unob}}(L)$. Namely, for any $L' \in \mathrm{Lag}(L)$, Lemma 2.2 states that $\psi \in \mathrm{Symp}_0(P,\Omega)$ such that $L' = \psi(L)$. Let $\mathbf{b} \in \mathcal{M}_{\mathrm{def,weak}}(L)$. Then $\psi_*\mathbf{b} \in \mathcal{M}_{\mathrm{def,weak}}(L')$, i.e., L' is unbostructed. Combining the argument in [5], we obtain the following:

Theorem 4.2. *Let $i : L \subset P$ be a relatively spin Lagrangian submanifold such that $i^* : \mathrm{H}^*(P;\mathbb{R}) \to \mathrm{H}^*(L;\mathbb{R})$ is surjective. Then $\mathrm{Lag}(L)/\mathrm{Ham}(P,\Omega)$ is Hausdorff with respect to the C^1-topology.*

Remark 4.2. In the case that $\mathrm{H}^2(P;\mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^2(L;\mathbb{Z}/2\mathbb{Z})$ is surjective, L is a relatively spin submanifold. In particular, the diagonal set $\Delta \subset M \times M$ satisfies the assumption in Theorem 4.1. Hence $\mathrm{Lag}(\Delta)/\mathrm{Ham}(M^{\times 2}, \tilde{\omega})$ is Hausdorff.

Sketch of the proof. We will show that the $\mathrm{Ham}(P,\Omega)$-orbits are closed in the C^1-topology. Then the Hausdorff property follows as in the proof of Theorem 1.2.

Firstly, we note that L is unobstructed after infinitesimal deformation by Corollary 3.4. Pick $\mathbf{b} \in \mathcal{M}_{\mathrm{def,weak}}(L)$. Suppose that there exists $\phi \in \mathrm{Ham}(P,\Omega)$ such that L and $L' = \phi(L)$ are sufficiently close to $L_* \in \mathrm{Lag}(L)$. Let $\mathcal{U}$ be as in the proof of Theorem 1.1. Then L, resp. L', correspond to a closed 1-form η, resp., η' on L_* such that $\| \eta \|_{C^1(\Omega^1(L_*))}, \| \eta' \|_{C^1(\Omega^1(L_*))} < \epsilon$. For a closed 1-form θ on L_* denote by $H_\theta \to L$ a flat line bundle corresponding to $[\theta] : \pi_1(L) \to \mathbb{R}$.

Theorem 3.2 also holds in the case of twisted by flat bundles on $L^{(i)}$. We use the flat line bundle $H_{\epsilon(\eta'-\eta)}$ on L and the trivial bundle $\underline{\mathbb{R}}$ on L'. (We identify L and L_* by the projection $L \subset U \to L_*$ and regard η, η' as closed 1-forms on L.) Thus we have

$$\mathrm{HF}^*\left((L', \mathbf{b}', \underline{\mathbb{R}}), (L, \mathbf{b}, H_{\epsilon(\eta'-\eta)}); \Lambda_{nov}\right) \cong \mathrm{HF}^*\left((L, \mathbf{b}, \underline{\mathbb{R}}), (L, \mathbf{b}, H_{\epsilon(\eta'-\eta)}); \Lambda_{nov}\right).$$

While the right hand side is described by ordinary cohomology of L with coefficients in $H_{\epsilon(\eta'-\eta)}$, the left hand side is described by Novikov cohomology with coefficients in $H_{\epsilon(\eta'-\eta)}$, cf. Theorem H, Theorem 24.5 in [4]. Then we can derive a contradiction in a similar way as in [5]. □

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References

[1] Y. Chekanov, *Lagrangian tori in a symplectic vector space and global symplectomorphisms*, Math. Z. (1996), **223**, 547-559.

[2] A. Floer, *Witten's complex and infinite dimensional Morse theory*, J. Differential Geom. **30** (1989), 207-221.

[3] K. Fukaya, *Floer homology and mirror symmetry, I*, Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds, AMS/IP Stud. Adv. Math. **23**, 15-43, Amer. Math. Soc., 2001.

[4] K. Fukaya, Y. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory -anomaly and obstruction-*, (2000, revised version 2006).

[5] K. Ono, *Floer-Novikov cohomology and the flux conjecture*, Geom. Funct. Anal. **16** (2006), 981-1020.

[6] K. Ono, *Development in symplectic Floer theory*, Proceedings of ICM, Madrid 2006, Vol. II, 1061-1082, Eur. Math. Soc. (2006).

[7] P. Seidel, *Graded Lagrangian submanifolds*, Bull. Soc. Math. France **128** (2000), 103-149.

[8] R. Thomas, *Moment maps, monodromy and mirror manifolds*, Symplectic Geometry and Mirror Symmetry, Edited by K. Fukaya, Y.-G. Oh, K. Ono and G. Tian, 467-498, World Scientific, 2001.

[9] R. Thomas and S. T. Yau, *Special Lagrangians, stable bundles and mean curvature flow,* Comm. Anal. Geom. **10** (2002), 1075-1113.

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Hamiltonian handleslides for Heegaard Floer homology

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ABSTRACT. A g-tuple of disjoint, linearly independent circles in a Riemann surface Σ of genus g determines a 'Heegaard torus' in its g-fold symmetric product. Changing the circles by a handleslide produces a new torus. It is proved that, for symplectic forms with certain properties, these two tori are Hamiltonian-isotopic Lagrangian submanifolds. This provides an alternative route to the handleslide-invariance of Ozsváth–Szabó's Heegaard Floer homology.

1. Introduction

1.1. Handlebodies and handleslides

Let Σ be a surface of genus $g \geq 1$. We can express Σ as the boundary ∂U of a 3-dimensional handlebody U by choosing g disjoint, embedded circles, $(\gamma_1, \ldots, \gamma_g)$, linearly independent in homology: each of these is filled so as to bound a disc in U. Conversely, the handlebody U determines the equivalence class of the g-tuple of attaching circles $(\gamma_1, \ldots, \gamma_g)$ under the equivalence relation generated by isotopies, permutations and *handleslides* (see, for instance, [5]).

Handleslides are not possible when $g = 1$ (the attaching circle for a genus 1 handlebody is unique up to isotopy), so from now on we shall assume that $g \geq 2$. A handleside is a replacement

$$(\gamma_1, \gamma_2, \ldots, \gamma_g) \rightsquigarrow (\gamma_0, \gamma_2, \ldots, \gamma_g),$$

where γ_0 is disjoint from $\gamma_2 \cup \cdots \cup \gamma_g$ and isotopic to another circle γ_0', disjoint from $\gamma_1 \cup \gamma_2$, such that $\gamma_0' \cup \gamma_1 \cup \gamma_2$ bounds an embedded pair of pants in Σ; see Figure 1. This condition implies that the circles $(\gamma_0, \gamma_2, \ldots, \gamma_g)$ are again linearly independent in homology, and hence determine a handlebody U'. There is a diffeomorphism $U \to U'$ acting as the identity on $\partial U = \Sigma = \partial U'$, since, once the $g-1$ curves $(\gamma_2, \ldots, \gamma_g)$ have been collapsed, γ_0 becomes isotopic to γ_1.

1.2. Heegaard tori

Fix, once and for all, a complex structure j on Σ. The g-fold symmetric product $\mathrm{Sym}^g(\Sigma)$ is then a complex manifold and hence also a differentiable manifold. In working

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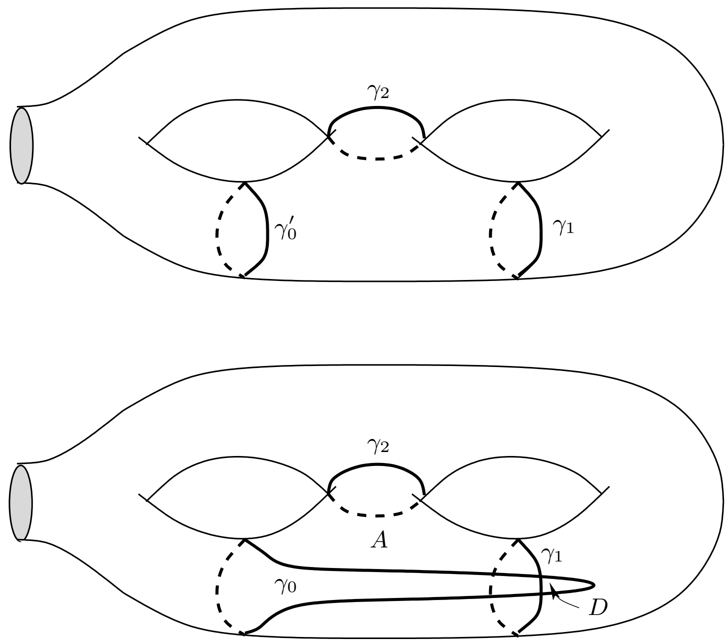

FIGURE 1. A handleslide in a punctured genus 2 surface. In the upper diagram, the three curves γ_0', γ_1 and γ_2 bound an embedded pair of pants. In the lower diagram, γ_0 intersects γ_1 in two points; this pattern of intersection is the one we shall use in the proof of our theorem.

with $\mathrm{Sym}^g(\Sigma)$ one should be aware that its differentiable structure depends on j but its diffeomorphism type does not. The g-tuple of disjoint attaching circles $(\gamma_1, \dots, \gamma_g)$ determines an embedded torus in $\mathrm{Sym}^g(\Sigma)$,

$$T_1 = \pi(\gamma_1 \times \gamma_2 \times \cdots \times \gamma_g),$$

its *Heegaard torus*. Here $\pi\colon \Sigma^{\times g} \to \mathrm{Sym}^g(\Sigma)$ is the quotient map; it restricts to an embedding on $\gamma_1 \times \gamma_2 \times \cdots \times \gamma_g$ since the circles γ_i are disjoint. Likewise, the g-tuple $(\gamma_0, \gamma_2, \dots, \gamma_g)$ determines a torus

$$T_0 = \pi(\gamma_0 \times \gamma_2 \times \cdots \times \gamma_g).$$

These tori are totally real, that is, their tangent spaces contain no complex lines; indeed, if we choose an area form α on Σ then the complement of the diagonal in $\mathrm{Sym}^g(\Sigma)$ has an induced symplectic form $\pi_*(\alpha^{\times g})$, and this form makes the two tori Lagrangian. Note, however, that the push-forward of a 2-form by a finite holomorphic map is not globally a smooth 2-form but rather a current, with singularities along the branch locus—in this case, the diagonal.

We now recall that for $n > 1$, one has $H^2(\mathrm{Sym}^n(\Sigma);\mathbb{Z}) \cong H^0(\Sigma;\mathbb{R}) \oplus \Lambda^2 H^1(\Sigma;\mathbb{Z})$ by an isomorphism equivariant under the actions of the mapping class group on the two sides. Write η for the class corresponding to $1 \in H^0(\Sigma;\mathbb{Z})$, or for its image in $H^2(\mathrm{Sym}^n(\Sigma);\mathbb{R})$ (this leaves a sign ambiguity which can be fixed by giving an alternative description: η is Poincaré dual to the divisor $\mathrm{Sym}^{g-1}(\Sigma)$, embedded by $D \mapsto p + D$ for some fixed p). When $g > 0$, write θ for (the image in real cohomology of) the generator of the cyclic group of classes in the summand $\Lambda^2 H^1(\Sigma;\mathbb{Z})$ invariant under the mapping class group. This time the sign ambiguity can be fixed by saying that θ is the pullback by the Abel–Jacobi map of an ample class (the theta-divisor) on the Jacobian. Suitable references include [1, 2].

Our first, preliminary result is as follows.

Proposition 1.1. *Suppose $(\gamma_0, \gamma_2, \ldots, \gamma_g)$ is a g-tuple of disjoint, linearly independent attaching circles, and $(\gamma_1, \gamma_2, \ldots, \gamma_g)$ another g-tuple differing from the first by a handleslide. For real numbers λ with sufficiently small absolute value, and arbitrary positive area forms α on Σ, there exist symplectic forms ω_λ on $\mathrm{Sym}^g(\Sigma)$, taming the complex structure $\mathrm{Sym}^g(j)$, such that*

(1) *ω_λ agrees with the product form $\pi_*(\alpha^{\times g})$ in neighbourhoods of the Heegaard tori T_0 and T_1; and*
(2) *ω_λ represents the class $\eta + \lambda\theta$.*

One can in fact take ω_λ to be Kähler. A short proof can be given using smoothing theory for currents, as in the author's earlier unpublished note [10] which has now been incorporated as the last section (7) of this paper. An immediate consequence of this smoothing theory (Corollary 7.2) is that there are forms satisfying the proposition in the class η. From this it is easy to deduce the result: one can perturb such a form by adding a small multiple of a closed $(1,1)$-form supported near the diagonal δ and Poincaré dual to δ. Since $PD[\delta] = \eta - \theta$,[1] these perturbed forms do the job.

This paper actually contains a second proof of the proposition, independent from that given in Section 7. The second proof is, however, valid only when $\lambda > 0$, and the forms it produces are not necessarily Kähler. It can be found in Section 4, subsumed in the proof of our main theorem, which we now proceed to state.

1.3. The main theorem

For a form ω_λ as in Proposition 1.1, the tori T_0 and T_1 are Lagrangian. One may ask whether they are isotopic through Lagrangians, or indeed Hamiltonian-isotopic. To make these into well-posed questions, we suppose that γ_0 and γ_1 intersect transversely in precisely two points, as in Figure 1. We also fix α. The answers to the questions are then independent of the particular form ω_λ and depend only on the parameter λ. Indeed, it follows from Moser's lemma and the convexity of the set of symplectic forms taming

[1] The tangent bundle to $\mathrm{Sym}^n(\Sigma)$ can be described algebro-geometrically as $\mathcal{O}_\delta(\delta)$. Hence $PD[\delta] = c_1(T\,\mathrm{Sym}^n(\Sigma)) = (n + 1 - g)\eta - \theta$ [1].

$\mathrm{Sym}^g(j)$ that, when such forms are cohomologous, they are related by symplectomorphisms supported away from $T_0 \cup T_1$.

We answer these questions when $\lambda > 0$. The answers depend on the areas $\int_D \alpha$ and $\int_A \alpha$ of the half-disc D enclosed by arcs in γ_0 and γ_1 and the the annular region A enclosed by γ_2 and the remaining portions of γ_0 and γ_1 (see Figure 1).

Theorem 1.2. *Suppose $(\gamma_0, \gamma_2, \dots, \gamma_g)$ is a g-tuple of disjoint, linearly independent attaching circles, and $(\gamma_1, \gamma_2, \dots, \gamma_g)$ another g-tuple differing from the first by a handleslide. Assume that $\gamma_0 \cap \gamma_1$ is a transverse intersection consisting of precisely two points. Fix a positive area form α on Σ, and let ω_λ be a Kähler form satisfying the conclusions of the last proposition. Suppose that λ is strictly positive. Then*

(1) *T_0 is Lagrangian-isotopic to T_1.*
(2) *T_0 is Hamiltonian-isotopic to T_1 if and only if $\int_D \alpha = (1+\lambda)\int_A \alpha - 2\lambda$.*
(3) *A Lagrangian isotopy (or indeed a Hamiltonian one, when the area constraint is satisfied) may be constructed as the product of the constant isotopy on the common factor $\gamma_3 \times \dots \times \gamma_g$ of the two tori and an isotopy in $\mathrm{Sym}^2(\Pi)$, where $\Pi \subset \Sigma$ is an (arbitrary) embedded pair of pants containing $\gamma_0 \cup \gamma_1 \cup \gamma_2$ in its interior.*

The proof of Lagrangian isotopy is constructive. The idea is to exploit the fact that γ_0 and γ_1 become isotopic in Σ', the surface obtained by surgering out γ_2. The genus 2 case contains the heart of the matter. When Σ has genus 2, the surgery is mimicked symplectically by associating with γ_2 a hypersurface $V \subset \mathrm{Sym}^2(\Sigma)$ which is the total space of an S^1-bundle $\rho\colon V \to \Sigma'$. The pullback by ρ of an area form on Σ' agrees with the symplectic form on $\mathrm{Sym}^2(\Sigma)$ restricted to V. The preimage under ρ of a Lagrangian isotopy between the images of γ_0 and γ_1 in Σ' is then a Lagrangian isotopy in $\mathrm{Sym}^2(\Sigma)$. One needs to be able to exert enough control over V to arrange that the isotopy begins at $\gamma_0 \times \gamma_2$ and ends at $\gamma_1 \times \gamma_2$.

Note that the forms used in the proof are *not* the perturbed, smoothed currents mentioned above, but rather, symplectic structures arising from presentations of symmetric products as symplectic sums. However, as already noted, the existence of Lagrangian or Hamiltonian isotopies is a function of α and λ and not of the particular form ω_λ.

Remark 1.1. It is an open question whether one can find a form making T_0 and T_1 into Hamiltonian-isotopic Lagrangians when $\lambda = 0$ and $\int_D \alpha = \int_A \alpha$. It is conceivable that we have here an instance of *symplectic fragility*, the phenomenon whereby non-isotopic Lagrangians become isotopic as soon as a perturbation parameter is switched on [14], though this seems rather unlikely since we shall find that the Floer-theoretic properties of (T_0, T_1) do not change as λ decreases to 0. In the gauge-theoretic interpretation of symmetric products as moduli spaces of vortices, the natural symplectic form represents (up to scale) a class $\eta + \lambda\theta$ where λ decreases to 0 as the 'stability parameter' tends to infinity [12].

We shall use Theorem 1.2 to re-prove the handleslide-invariance of Ozsváth–Szabó's Heegaard Floer homology [9, Section 9] (the argument still draws on finiteness lemmas from [9]). The precise statement uses notions from [9] which we do not recapitulate here.

Corollary 1.3. *Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a pointed Heegaard diagram, weakly admissible for a* Spin^c*-structure $\mathfrak{s}$, and suppose that $\boldsymbol{\alpha}'$ differs from $\boldsymbol{\alpha}$ by a handleslide. Suppose that the circle that is being replaced intersects its replacement non-trivially in the pattern of Theorem 1.2. Assume that z lies outside the handlesliding region. Then there is a canonical isomorphism $HF^+(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}) \to HF^+(\boldsymbol{\alpha}', \boldsymbol{\beta}, \mathfrak{s})$ between the Heegaard Floer homology groups for the* Spin^c*-structure $\mathfrak{s}$, and similarly for HF^∞, HF^- and $\widehat{HF}$.*

Referring to Figure 1, the 'handlesliding region' means the union of the region bounded by γ_2 and arcs in γ_0 and γ_1 and that bounded by arcs in γ_0 and γ_1.

Organisation of this paper is as follows. In Section 2, we review the relationship between degenerations of symplectic manifolds and the symplectic sum operation; this is then applied to symmetric products (Section 3), and employed to construct the Lagrangian and Hamiltonian isotopies of the theorem (Section 4). Section 5 proves the Hamiltonian non-isotopy clause, and Section 6 covers the application to Heegaard Floer homology. The symplectic forms used up to that point tame the natural complex structure on $\mathrm{Sym}^g(\Sigma)$ but are not actually Kähler. It has already been mentioned that in Section 7 we give a different construction, by smoothing currents, which does produce Kähler forms and has some independent interest.

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2. Degeneration and symplectic sums

In this section we review the notion of symplectic sum and its relation to degenerations of symplectic manifolds [4, 6]. So as to simplify the definitions (slightly) we restrict the discussion to complex manifolds.

Definition 2.1. If M is a complex n-manifold, a *nodal degeneration* of M is a pair (E, π) where E is a complex $(n+1)$ manifold and $\pi\colon E \to \mathbb{C}$ a holomorphic map which is a topological fibre bundle over $\mathbb{C}^*$, with fibre $E_1 = M$ (we write E_t for $\pi^{-1}(t)$). Thus the critical locus $C = \mathrm{crit}(\pi)$ is contained in the zero-fibre E_0. We suppose that this locus is smooth and $(n-1)$-dimensional, and that the (complex) Hessian on $N_{C/E}$ is everywhere non-degenerate.

As a complex space, E_0 has a normal crossing along C: locally near a point of C, E_0 is equivalent to a neighbourhood of the origin in $\mathbb{C}^{n-1} \times \{z_1 z_2 = 0\} \subset \mathbb{C}^{n-1} \times \mathbb{C}^2$. Let

$n\colon \tilde{E}_0 \to E_0$ be the normalisation of E_0. The normalisation is an intrinsic construction but, put simply, it is the complex manifold obtained by replacing each patch $\mathbb{C}^{n-1} \times \{z_1 z_2 = 0\}$ by $\mathbb{C}^{n-1} \times (\mathbb{C} \amalg \mathbb{C})$. The preimage $n^{-1}(C) \subset \tilde{E}_0$ is the disjoint union of two divisors Z_1 and Z_2 which are identified with one another via n. Their normal bundles are dually paired with one another by means of the Hessian $D^2\pi$ on $N_{C/E}$. Choose closed tubular neighbourhoods U_1 and U_2 in $\tilde{E}_0$ for Z_1 and Z_2 respectively, and put $V_i = \partial U_i$, so that V_i is an S^1-bundle over Z_i. There is an orientation-reversing diffeomorphism $\sigma\colon V_1 \to V_2$, which arises because because we can think of V_1 as the equator in the $\mathbb{CP}^1$-bundle $\mathbb{P}(N_1 \oplus \mathcal{O}) \to Z_1 = C$, and V_2 as the equator in $\mathbb{P}(\mathcal{O} \oplus N_2^*) \to Z_2 = C$; but using the Hessian we have

$$\mathbb{P}(N_1 \oplus \mathcal{O}) = \mathbb{P}((N_1 \otimes N_2^*) \oplus N_2^*) \cong \mathbb{P}(\mathcal{O} \oplus N_2^*).$$

This discussion makes sense symplectically. If we are given a symplectic form Ω on E, taming the complex structure, the manifold $\tilde{E}_0$ inherits a symplectic form ω for which $n_*(\omega|_{Z_1}) = n_*(\omega|_{Z_2})$. We are thus in a position to form Gompf's *symplectic sum*

$$\#_{Z_1 \sim Z_2} \tilde{E}_0$$

as in [4], the symplectic manifold obtained from $\tilde{E}_0 \setminus (\operatorname{int} U_1 \cup \operatorname{int} U_2)$ by gluing V_1 to V_2 via σ.

Proposition 2.1. *The fibre $M = E_1$ of a nodal degeneration (E, π), when equipped with the restriction of a Kähler form Ω on E, is symplectomorphic to the symplectic sum $\#_{Z_1 \sim Z_2} \tilde{E}_0$.*

Sketch of proof. We refer to [6] for a complete treatment; here we aim mainly to draw attention to an important aspect of the geometry of degenerations, their *vanishing cycles*.

The vanishing cycle $V \subset M$ is the set of points x such that symplectic parallel transport $M = E_1 \to E_t$ over the ray $[t, 1] \subset \mathbb{C}$ lands in the critical set C in the limit $t \to 0^+$. The limiting parallel transport defines an S^1-bundle $\rho\colon V \to C$, and the crucial point for us will be that

$$\Omega|_V = \rho^*(\Omega|_C).$$

Correspondingly, in the symplectic sum $\#_{Z_1 \sim Z_2} \tilde{E}_0$, let V' denote the common image of V_1 and V_2. Since it arises as the boundary of a tubular neighbourhood of C, it comes equipped with an S^1-bundle $\rho'\colon V' \to C$.

Notice that $(\#_{Z_1 \sim Z_2} \tilde{E}_0) \setminus V'$ is naturally identified symplectically with $E_0 \setminus C$. On the other hand, symplectic parallel transport into E_0 defines a symplectomorphism $M \setminus V \to E_0 \setminus C$. Combining these observations we find a symplectomorphism $M \setminus V \cong (\#_{Z_1 \sim Z_2} \tilde{E}_0) \setminus V'$. But both V and V' are coisotropic submanifolds in symplectic manifolds (in fact, they are S^1-bundles over a common symplectic manifold). There is a diffeomorphism $V \to V'$ covering the identity map on C, and by the coisotropic neighbourhood theorem, this extends to a symplectomorphism between neighbourhoods of V and V'. It remains to see that these can be made compatible with the identification $M \setminus V \cong (\#_{Z_1 \sim Z_2} \tilde{E}_0) \setminus V'$.

This requires closer inspection of the local model for the symplectic sum, and we shall not give the details here (see [6]; also compare [11, Section 2]). □

3. Symmetric products as symplectic sums

Let $\gamma \subset \Sigma$ be a simple closed curve, and let Σ_γ be the surface obtained by excising a tubular neighbourhood of γ and gluing in a pair of discs. Let p and q be points in the respective interiors of those discs. Fix complex structures on Σ and Σ_γ, and an integer $n \geq 1$.

Construction 3.1. Define two maps

$$i_p, i_q \colon \operatorname{Sym}^{n-1}(\Sigma_\gamma) \to \operatorname{Sym}^n(\Sigma_\gamma)$$

by $i_p(D) = p + D$, $i_q(D) = q + D$. Denote by $\mathbf{S}_n(\Sigma_\gamma)$ the complex blow-up of $\operatorname{Sym}^n(\Sigma_\gamma)$ along the locus $p + q + \operatorname{Sym}^{n-2}(\Sigma_\gamma)$. Let

$$\tilde{\imath}_p, \tilde{\imath}_q \colon \operatorname{Sym}^{n-1}(\Sigma_\gamma) \to \mathbf{S}_n(\Sigma_\gamma)$$

be the natural lifts of i_p and i_q. The images $Z_p = \operatorname{im} \tilde{\imath}_p$ and $Z_q = \operatorname{im} \tilde{\imath}_q$ are disjoint. Let $\tau = \tilde{\imath}_p \circ \tilde{\imath}_q^{-1} \colon Z_q \to Z_p$. There is a natural isomorphism $\tau^* N_{Z_p/\mathbf{S}_n(\Sigma_\gamma)} \cong N^*_{Z_q/\mathbf{S}_n(\Sigma_\gamma)}$ (see [11, Section 3]). Thus τ identifies the divisors Z_p and Z_q, and under this identification, their normal bundles are dual. We may therefore form the symplectic sum $\#_{Z_p \sim Z_q} \mathbf{S}_n(\Sigma_\gamma)$ (as a smooth manifold; we will say more about symplectic forms presently).

Proposition 3.1. *There is a diffeomorphism*

$$\phi \colon \#_{Z_p \sim Z_q} \mathbf{S}_n(\Sigma_\gamma) \cong \operatorname{Sym}^n(\Sigma),$$

canonical up to isotopy.

Proof. The diffeomorphism-type of a symmetric product of a Riemann surface does not depend on its complex structure. Moreover, because the space of complex structures is simply connected, there are canonical diffeomorphisms between them (up to isotopy). Thus we can choose complex structures on Σ and Σ_γ as we wish.

To prove the proposition, we invoke Proposition 2.1. What we are asserting is that there is a nodal degeneration of $\operatorname{Sym}^n(\Sigma)$ to a complex space whose normalisation is $\mathbf{S}_n(\Sigma_\gamma)$, in which Z_p and Z_q are the two preimages of the normal crossing divisor, and that the pairing of normal bundles is induced by the Hessian along the critical locus of the degeneration.

Let $E \to \mathbb{C}$ be a nodal degeneration of Σ, i.e. a holomorphic Lefschetz fibration such that $E_1 = \Sigma$, with a single critical point lying over 0. We arrange that the vanishing cycle, taken along the vanishing path $[0, 1]$, is (the isotopy class of) γ. One can then form an associated family, $\operatorname{Hilb}^n(\pi) \to \mathbb{C}$, the relative Hilbert scheme of n points, as in Donaldson–Smith [3]. Their crucial observation (later re-proved by Ran [13]) is that this space is globally smooth; indeed, it is a nodal degeneration of its smooth fibre. There is a natural morphism $\operatorname{Hilb}^n(\pi) \to \operatorname{Sym}^n(\pi)$ to the relative symmetric product, whose restriction to $\mathbb{C}^*$ is biholomorphic onto the restriction of $\mathrm{Sym}^n(\pi)$ to $\mathbb{C}^*$; hence $\mathrm{Hilb}^n(\pi)$ is a nodal degeneration of $\mathrm{Sym}^n(\Sigma)$.

The structure of the relative Hilbert scheme was described in elementary terms in [11]; in particular, it was explained that the normalisation of the zero-fibre $\mathrm{Hilb}^n(E_0)$ is precisely $\mathbf{S}_n(\widetilde{E}_0)$ where $\widetilde{E}_0 \to E_0$ is the normalisation of the zero-fibre of $E \to \mathbb{C}$. The two distinguished divisors inside it are the divisors Z_p and Z_q described in the discussion above (with p and q the points of $\widetilde{E}_0$ lying over the node of E_0), so $\mathrm{Hilb}^n(E_0)$ is obtained by identifying Z_p with Z_q via τ. Thus our result follows from the general theory of Section 2. □

A case to keep in mind is the symmetric square, $\mathrm{Sym}^2(\Sigma)$. In this case, $\mathbf{S}_2(\Sigma_\gamma)$ is the blow-up of $\mathrm{Sym}^2(\Sigma_\gamma)$ at the point $p+q$. The two divisors Z_p and Z_q are disjoint copies of Σ_γ. If γ is a separating curve, so that Σ_γ is a disjoint union $\Sigma_1 \amalg \Sigma_2$, the assertion is that $\mathrm{Sym}^2(\Sigma)$ is the fibre sum

$$\mathrm{Sym}^2(\Sigma_1)\#_{\Sigma_1}(\widetilde{\Sigma_1 \times \Sigma_2})\#_{\Sigma_2}\mathrm{Sym}^2(\Sigma_2).$$

Here $\widetilde{\Sigma_1 \times \Sigma_2}$ is the blow-up of $\Sigma_1 \times \Sigma_2$ at (p,q); the proper transforms of the factors in the product give embeddings of Σ_1 and Σ_2 into the blow-up, each of self-intersection -1. On the other hand, $\mathrm{Sym}^2(\Sigma_1)$ contains a copy of Σ_1 of self-intersection $+1$, embedded by the map $x \mapsto p+x$. Likewise, $\mathrm{Sym}^2(\Sigma_2)$ contains a copy of Σ_2 of self-intersection $+1$, embedded by the map $x \mapsto q+x$. The fibre sums $\#_{\Sigma_1}$ and $\#_{\Sigma_2}$ are taken along these two pairs of surfaces. (Note: The manifold appearing here is written as a sum of three different manifolds along two pairs of surfaces, whereas the proposition expresses it as a self-fibre sum of a disconnected manifold along a disconnected surface.)

Remark 3.1. This remark is due to R. Fintushel.[2] When Σ_2 has genus 1, the description can be simplified because the second fibre sum has no topological effect. Thus, if we write $\Sigma = \Sigma' \# T^2$ (connected sum) we have

$$\mathrm{Sym}^2(\Sigma) \cong \mathrm{Sym}^2(\Sigma')\#_{\Sigma'}(\widetilde{\Sigma' \times T^2}).$$

Indeed, we know that $\mathrm{Sym}^2(\Sigma) \cong \mathrm{Sym}^2(\Sigma')\#_{\Sigma'}(\widetilde{\Sigma' \times T^2})\#\,\mathrm{Sym}^2(T^2)$. The summand $\mathrm{Sym}^2(T^2)$ is the total space of a non-trivial S^2-bundle over T^2 (the bundle projection is the Abel–Jacobi map). To perform the second fibre sum, we remove the tubular neighbourhood of a square -1 torus in $\mathrm{Sym}^2(\Sigma)'\#(\widetilde{\Sigma' \times T^2})$ (the proper transform of $T^2 \times \{\text{pt.}\}$ in the second summand). We glue in the complement of a tubular neighbourhood of a square $+1$ section s_0 of $\mathrm{Sym}^2(T^2) \to T^2$. This complement is a tubular neighbourhood of another section s_∞, which must have square -1 (since the fact that $[s_\infty]\cdot[s_0] = 0$ forces $[s_\infty] = [s_0] - [\text{fibre}]$). Thus we glue into $\mathrm{Sym}^2(\Sigma)'\#(\widetilde{\Sigma' \times T^2})$ the same piece that we removed, without changing the gluing.

[2]The conclusion was also known to I. Baykur. Any mistakes in the argument are the author's responsibility.

We now describe how the symplectic sum description affects the relevant cohomology classes, adding subscripts to the notation to track which surface we are considering.

Lemma 3.2. *Assume Σ_γ is connected. Let $c_\lambda \in H^2(\mathbf{S}_n(\Sigma_\gamma);\mathbb{R})$ be the pullback of $\eta_{\Sigma_\gamma} + \lambda\theta_{\Sigma_\gamma}$ from $\mathrm{Sym}^n(\Sigma_\gamma)$ to its blow-up, minus λ times the class Poincaré dual to the exceptional divisor. Under the diffeomorphism ϕ of the previous proposition, c_λ pulls back to $\eta_\Sigma + \lambda\theta_\Sigma$.*

Closely related results were obtained in [11], notably Proposition 3.14.

The relevance of c_λ is that it will be the class of our symplectic form. See [7, 17] for accounts of blowing up symplectic or Kähler manifolds.

Proof. We shall give a direct proof in a 'prototypical' case, then explain how to obtain the general result from this.

Consider $\mathrm{Sym}^2(T^2)$, thought of as $\#_{\mathbb{P}^1\sim\mathbb{P}^1}\widetilde{\mathbb{P}}^2$ (symplectic sum along two square-zero 2-spheres). One has $H_2(\widetilde{\mathbb{P}}^2;\mathbb{Z}) = \mathbb{Z}^2$, the generators being the classes e of the exceptional divisor and s of the proper transform of a (generic) line in $\mathbb{P}^2$. The fibre sum description corresponds to a nodal degeneration $\mathrm{Hilb}^2_{\mathbb{C}}(E) \to \mathbb{C}$; the homology cycles in $\widetilde{\mathbb{P}}^2$ project to cycles in the central fibre of $\mathrm{Hilb}^2_{\mathbb{C}}(E)$, and (by thinking of the symplectic sum construction) one can see explicitly that these projected cycles are homologous in $\mathrm{Hilb}^2_{\mathbb{C}}(E)$ to cycles in $\mathrm{Sym}^2(T^2)$.

Indeed, e intersects each of the two square-zero $\mathbb{P}^1$s in a point, and so deleting a neighbourhood of $\mathbb{P}^1 \amalg \mathbb{P}^1$ has the effect of removing two discs from e; in the symplectic sum, the two boundary circles of this 2-holed sphere are identified to form a torus $\tilde{e}$ in $\#_{\mathbb{P}^1\sim\mathbb{P}^1}\widetilde{\mathbb{P}}^2 = \mathrm{Sym}^2(T^2)$, of square -1, homologous in $\mathrm{Hilb}^2_{\mathbb{C}}(E)$ to the projection of e. Starting from s, one similarly gets a square $+1$ torus in $\mathrm{Sym}^2(T^2)$. Note that both $\tilde{e}$ and $\tilde{s}$ can be thought of as sections of the Abel–Jacobi fibration $\mathrm{Sym}^2(T^2) \to T^2$, $[x,y] \mapsto x+y$ ($x+y$ means the sum under the group law of an elliptic curve). One now sees that $\tilde{e}$ is dual to θ_Σ, $\tilde{s}$ to η_Σ. Thus in this case, the symplectic class $(\eta_{\Sigma_\gamma} + \theta_{\Sigma_\gamma}) + \lambda e$ does indeed correspond to $\eta_\Sigma + \lambda\theta_\Sigma$ (but here $\theta_{\Sigma_\gamma} = 0$).

To obtain the general formula, it suffices to do so when $n = 2$. Indeed, the construction of the degeneration of Σ, and hence of its relative Hilbert scheme, is natural under diffeomorphisms of Σ which act trivially near γ. Thus $\phi^* c_\lambda$ has to be invariant under the corresponding subgroup of the mapping class group. It must also restrict to the $n = 2$ version of $\eta + \lambda\theta$ when restricted to $(n-2)z + \mathrm{Sym}^2(\Sigma)$ (where z is a basepoint). Bearing in mind that $H^2(\mathrm{Sym}^n(\Sigma);\mathbb{R}) = \mathbb{R} \oplus \Lambda^2 H^1(\Sigma;\mathbb{R})$ independent of $n \geq 2$, and that θ and η restrict to $(n-2)z + \mathrm{Sym}^2(\Sigma)$ in the obvious way, this shows that it is enough to prove the formula when $n = 2$.

Observe that we can embed a one-holed torus T' into Σ as a neighbourhood of γ. The restriction of $\phi^* c_\lambda$ to $\mathrm{Sym}^2(T') \subset \mathrm{Sym}^2(\Sigma)$ is then independent of the topology of Σ. From this observation and the case worked out above one obtains the $n = 2$ result. $\square$

When γ disconnects Σ, the lemma is still true but rather trivial. In fact, in this case the weight of the blow-up makes no difference to the induced class c_λ on $\mathrm{Sym}^n(\Sigma)$.

These assertions are simple algebraic consequences of the naturality of the construction of $\mathbf{S}_2(\Sigma_\gamma)$ from Σ under diffeomorphisms of Σ supported away from γ.

4. Lagrangian isotopies

We prove statements (1) and (3) of Theorem 1.2. The argument also contains one of our two proofs of Proposition 1.1, though this one requires $\lambda > 0$ and does not produce Kähler forms.

Step 1a: Reduction to genus 2.

If Σ has genus $g \geq 3$, we can find a simple closed curve Γ separating Σ into a genus 2 part containing γ_0, γ_1 and γ_2, and a genus $g-2$ part containing $\gamma_3 \cup \cdots \cup \gamma_g$. Thus $\Sigma = \Sigma' \# \Sigma''$, where Σ' has genus 2 and contains the handleslide. Fix closed discs $D' \subset \Sigma'$ and $D'' \subset \Sigma''$ so that $\Sigma = (\Sigma' \setminus \mathrm{int}(D')) \cup_\Gamma (\Sigma'' \setminus \mathrm{int}(D''))$, and points $p \in \mathrm{int}(D')$, $q \in \mathrm{int}(D')$.

The region of $\mathrm{Sym}^g(\Sigma)$ of interest to us is the open set $\mathrm{Sym}^2(\Sigma' \setminus D') \times \mathrm{Sym}^{g-2}(\Sigma'' \setminus D'')$. We claim that there are Kähler forms on $\mathrm{Sym}^g(\Sigma)$ which restrict to this region as the sum of forms pulled back from the two factors. Once this claim is established, it will suffice to prove the theorem for a genus 2 surface. However, the reduction will not be perfect: the genus 2 surface has a puncture (far from the handleslide region), and we have to use the symplectic forms produced in the course of proving the claim.

Lemma 4.1. *For any $\lambda \geq 0$, there are symplectic (or indeed Kähler) forms ξ on $\mathrm{Sym}^g(\Sigma' \amalg \Sigma'')$, representing $\eta + \lambda\theta$, such that $i_p^*\xi = i_q^*\xi$. Moreover, ξ restricts to each connected component $\mathrm{Sym}^k(\Sigma') \times \mathrm{Sym}^{g-k}(\Sigma'')$ as the sum of two forms pulled back from the factors.*

Proof. Note that $\eta + \lambda\theta$ is a Kähler class (indeed, η is ample and θ is the pullback by the Abel–Jacobi map of an ample class on the Jacobian). Take a Kähler form κ_g' on $\mathrm{Sym}^g(\Sigma')$ representing $\eta + \lambda\theta$, and let κ_k' be its restriction to $\mathrm{Sym}^k(\Sigma')$, embedded by $D \mapsto (g-k)p + D$. Similarly, take a Kähler form κ_g'' on $\mathrm{Sym}^g(\Sigma'')$ representing $\eta + \lambda\theta$, and let κ_k'' be its restriction to $\mathrm{Sym}^k(\Sigma'')$, embedded by $D \mapsto (g-k)q + D$. Define ξ, on the connected component $\mathrm{Sym}^k(\Sigma') \times \mathrm{Sym}^{g-k}(\Sigma'')$, by $\kappa_{g-k}' \oplus \kappa_k''$. $\square$

Given such a form ξ, pull it back to $\mathbf{S}_g(\Sigma' \cup \Sigma'')$ (the blow-up of $\mathrm{Sym}^g(\Sigma' \cup \Sigma'')$ along $p + q + \mathrm{Sym}^{g-2}(\Sigma' \amalg \Sigma'')$) and add a closed form supported near the exceptional divisor so as to obtain a symplectic form ξ_λ. The weight of the blow-up is unimportant, because of the remark following the proof of Lemma 3.2. We may assume that $\tilde{i}_p^*\xi_\lambda = \tilde{i}_q^*\xi_\lambda$, so that ξ_λ may be used in forming the symplectic sum $\#_{Z_p \sim Z_q} \mathbf{S}_g(\Sigma' \amalg \Sigma'')$.

Now, there is a diffeomorphism $f\colon \mathrm{Sym}^g(\Sigma) \to \#_{Z_p \sim Z_q} \mathbf{S}_g(\Sigma' \amalg \Sigma'')$ (canonical up to isotopy), and thus $\omega := f^*\xi_\lambda$ is a symplectic structure on $\mathrm{Sym}^n(\Sigma)$. In setting up f, we need to fix tubular neighbourhoods of U_p of Z_p and U_q of Z_q. It is clear that we can choose U_p and U_q to be disjoint from the open subset $\mathrm{Sym}^2(\Sigma' \setminus D') \times \mathrm{Sym}^{g-2}(\Sigma'' \setminus D)$ of $\mathbf{S}_g(\Sigma' \amalg \Sigma'')$, and hence we can ensure that f restricts to this subset as the natural inclusion into $\mathrm{Sym}^g(\Sigma)$. With this duly arranged, the symplectic form ω fulfils our claim.

Step 1b: Completing the reduction to genus 2. In this step we complete the reduction to genus 2 by showing that we can choose *any* j-positive symplectic form representing the class $\eta + \lambda\theta$ on $\mathrm{Sym}^2(\Sigma')$. As things stand, our form is constrained to be the restriction to $(g-2)p + \mathrm{Sym}^2(\Sigma')$ of a j-positive symplectic form κ_g' on $\mathrm{Sym}^g(\Sigma')$. By an inductive argument, it will suffice to show that we can choose the Kähler form freely (within our preferred cohomology class) on the divisor $p + \mathrm{Sym}^{g-1}(\Sigma')$. Thus we have a j-positive symplectic form on a manifold M^{2n} and a codimension-2 j-holomorphic submanifold H^{2n-2}. The boundary of a tubular neighbourhood is a contact type hypersurface (convex, as seen from the inside). But, by an application of Gray's stability theorem—left to the reader—one can always isotope the symplectic form so that its restriction to H is a prescribed j-positive symplectic form. This does the trick.

Step 2: The (punctured) genus 2 case. Now assume Σ' is closed and has genus 2, and has a basepoint p far from any of the γ_i. We have to prove the theorem for $\Sigma' \setminus \mathrm{nd}(p)$.

Write T for the torus obtained by surgering out γ_2. The 'scar' left by the surgery is a pair of discs in T, containing points r and s in their interiors. In Σ', the surgery is done in a region disjoint from the curves γ_0 and γ_1, which therefore have images $\bar{\gamma}_0$ and $\bar{\gamma}_1$ in T.

At this point we should pin down the choice of j-positive symplectic form κ on $\mathrm{Sym}^2(\Sigma')$. We choose it to be a symplectic structure arising as the pullback via a diffeomorphism $\phi\colon \mathrm{Sym}^2(\Sigma') \cong \#_{Z_r \sim Z_s} \mathbf{S}_2(T)$ of a Kähler form on $\mathbf{S}_2(T)$. By Lemma 3.2, we can assume that it represents the cohomology class $\eta + \lambda\theta$.

More specifically, we take a Kähler form on $\mathrm{Sym}^2(T)$ representing $\eta + \lambda\theta$, and form its ordinary blow up in such a way that the exceptional divisor has weight λ. We have to make sure that the induced area forms on Z_r and Z_s agree with one another. For this, notice that, by the symplectic neighbourhood theorem, Z_r (say) has a neighbourhood of form $T \times D^2(0;R)$, with a product symplectic form $\varepsilon = \beta + \omega_C$. Think of the projection $p\colon T \times D^2(0;R) \to D(0;R)$ as a symplectic fibre bundle. By Thurston's patching argument [7, Chapter 6], we can replace the original symplectic form by a new closed 2-form ε', symplectic on the fibres of p, agreeing with the ε over the annulus $R/2 < |z| < R$, such that $\varepsilon'|_{p^{-1}(0)}$ is a freely chosen positive area form β'. For suitable 2-forms ζ on $D(0;R)$, supported in $D(0;R/2)$, the form $\varepsilon' + p^*(\omega_{\mathbb{C}} + \zeta)$, will then be symplectic. Thus, replacing ε by $\varepsilon' + p^*(\omega_{\mathbb{C}} + \zeta)$ near Z_r, we can adjust arrange that the forms on Z_r and Z_s agree.

We also need to be more precise about the symplectic sum and the diffeomorphism ϕ. To do this, we need to construct suitable tubular neighbourhoods of Z_r and Z_s. Given a compact subset $K \subset \Sigma' \setminus \{r, s\}$, there is a natural framing of the restricted normal bundle $N_{Z_r}|_{r+K}$ (more precisely, an identification with the trivial bundle with fibre $T_r\Sigma'$). Thus we can construct a symplectically trivial tubular neighbourhood of Z_r over the subset $r + K \subset Z_r$. Similarly for Z_s.

The symplectic sum description provides a hypersurface $V \subset \mathrm{Sym}^2(\Sigma')$—the vanishing cycle of the degeneration—whose isotropic foliation is by circles, and whose space of isotropic leaves is identified with T.[3] Thus there is an S^1-bundle $\rho\colon V \to T$ (topologically trivial, in fact) such that $\kappa|_V = \rho^*\beta$ for an area form β on T.

We want to make sure that $\rho^{-1}(K) = \gamma_2 + K \subset \mathrm{Sym}^2(\Sigma)$, and moreover, that $\rho(x+y) = x$ when $x \in K$, $y \in \gamma_2$. For this we have to set up the diffeomorphism ϕ correctly. We have already set up tubular neighbourhoods of $r+K$ and $s+K$; points $x+k$ in the boundary of the tubular neighbourhood (with $k \in K$) should map under ϕ to $x+k \in \mathrm{Sym}^2(\Sigma')$. This is a matter of smooth rather than symplectic topology; it can be arranged by the method of [11, Lemma 3.16], for instance. Note furthermore that, because of the way we have set up the tubular neighbourhoods of Z_r and Z_s, we end up with a symplectic form κ on $\mathrm{Sym}^2(\Sigma)$ which is product-like in $K \times \mathrm{nd}(\gamma_2)$.

Remark 4.1. Our symplectic forms on $\mathrm{Sym}^g(\Sigma)$ are not necessarily Kähler; the symplectic sum operation does not in general preserve the Kähler category. However, it is easy to see that they can be taken to tame the natural complex structure $\mathrm{Sym}^n(j)$.

Step 3: Lagrangian isotopy. Take an isotopy $\{\bar{\gamma}_t\}_{t\in[0,1]}$ of circles in T, from $\bar{\gamma}_0$ to $\bar{\gamma}_1$. It is, of course, a Lagrangian isotopy with respect to β. The preimages $\rho^{-1}(\bar{\gamma}_t)$ are tori in $\mathrm{Sym}^2(\Sigma')$. They are Lagrangian, because if u and v are tangent to $\rho^{-1}(\bar{\gamma}_t)$ at a point $\rho^{-1}(z)$ then $\kappa(u,v) = \beta(\rho_* u, \rho_* v) = 0$. Moreover, we set things up in Step 2 so that $\rho^{-1}(\bar{\gamma}_i) = \gamma_i \times \gamma_2$ for $i = 0,1$. Thus, we have a Lagrangian isotopy from $\gamma_0 \times \gamma_2$ to $\gamma_1 \times \gamma_2$. Notice that this isotopy stays inside a compact subset of $\mathrm{Sym}^2(\Sigma' \setminus \{p\})$, so it is again compatible with step 1.

This completes the proof of clauses (1) and (3) of Theorem 1.2.

Step 4: Hamiltonian isotopy. We begin the proof of statement (2) of Theorem 1.2.

The difference between Lagrangian and Hamiltonian isotopy is measured by the *flux* [7]. Given a Lagrangian isotopy $\{L_t\}$, its (infinitesimal) flux at time t is a class $a_t \in H^1(L_t;\mathbb{R})$ which can be obtained by extending the vector field generating the isotopy to a globally-defined symplectic vector field, dualising this to get a closed 1-form, and taking its cohomology class restricted to L_t. For the isotopy in Step 3, one checks that the flux a_t is equal to $\rho^* b_t$, where b_t is the flux of the isotopy $\{\bar{\gamma}_t\}$ in T with respect to β. Thus our Lagrangian isotopy in $\mathrm{Sym}^2(\Sigma')$ is Hamiltonian if and only if $\{\bar{\gamma}_t\}$ is a Hamiltonian isotopy in T. Now, the complement $T \setminus (\bar{\gamma}_0 \cup \bar{\gamma}_1)$ has three components, of which two are homeomorphic to the open disc. The only obstruction to making $\{\bar{\gamma}_t\}$ Hamiltonian is that the two disc-components have equal β-area.

We have now proved the existence of Hamiltonian isotopies under the condition that two discs have the same area with respect to the area form β on the surgered surface. However, the stated criterion concerned areas with respect to the area form α on Σ, and we need to relate these conditions. In stating the theorem, we noted that the existence of a Hamiltonian isotopy depends only on λ and α (and, *a priori*, on j). The problem

[3] The space of isotropic leaves of a (coisotropic) vanishing cycle V is naturally identified with the singular locus in the central fibre of the degeneration. The quotient map to the leaf-space corresponds to the limiting parallel transport map ρ from Proposition 2.1.

has an extra symmetry, however: we can apply self-diffeomorphisms to Σ, pulling back j, provided that they act trivially on $\gamma_0 \cup \gamma_1 \cup \gamma_2$.

Lemma 4.2. *The only invariants of area forms α under such diffeomorphisms are the areas of A, D and $\Sigma \setminus (A \cup D)$.*

Proof. Given area-forms α_0 and α_1, we can certainly find a self-diffeomorphism ψ, trivial along $\Gamma := \gamma_0 \cup \gamma_1 \cup \gamma_2$, such that $\psi^*\alpha_0 = \alpha_1$ in a neighbourhood of Γ. Now replace α_0 by $\psi^*\alpha_0$ and apply Moser's deformation argument to the path $\alpha_t = (1-t)\alpha_0 + t\alpha_1$. □

We thus see that the existence of Hamiltonian isotopies depends on α only through the areas of A and D (the area of their complement is clearly irrelevant). Our construction can be interpreted, then, as saying that there is a function $f(\lambda, \int_D \alpha)$ such that a Hamiltonian isotopy exists if $\int_A \alpha = f(\lambda, \int_D \alpha)$. In Section 5 we shall show that a Hamiltonian isotopy can *only* exist if $\int_A \alpha = (1+\lambda)^{-1}(\int_D \alpha + 2\lambda)$, which must therefore coincide with $f(\lambda, \int_D \alpha)$.

5. Hamiltonian non-isotopy

In this section we complete the proof of Theorem 1.2 by showing that the tori T_0 and T_1 can only be Hamiltonian-isotopic when $\int_D \alpha$ is related in a precise way to $\int_A \alpha$.

We do this by showing that if the area constraint fails there is a discrepancy between the ranks of the Lagrangian Floer homology groups $HF_*(T_0, T_0)$ and $HF_*(T_0, T_1)$. These would be isomorphic if a Hamiltonian isotopy existed. We work over the universal Novikov field $\Lambda_{\mathbb{Z}/2}$ of the field $\mathbb{Z}/2$, i.e., the ring of formal 'series' $\sum_{r\in\mathbb{R}} a(r)t^r$ where $a\colon \mathbb{R} \to \mathbb{Z}/2$ is a function such that $\operatorname{supp}(a) \cap (-\infty, c]$ is finite for all c. This coefficient ring is used to record the areas of holomorphic discs. The well-definedness and invariance of Floer homology for monotone Lagrangians of minimal Maslov index 2 is not automatic, but in the case of Heegaard tori it follows from a cancellation theorem from [9] (Theorem 3.15) alongside the transversality argument outlined in [8].

The self-Floer homology of T_0 is as large as it conceivably could be:

Lemma 5.1. $HF_*(T_0, T_0) \cong H_*(T_0; \Lambda_{\mathbb{Z}/2})$ *for any form ω_λ as in Proposition 1.1.*

Proof. The relevant moduli spaces are computed by Ozsváth–Szabó in [9, Lemma 9.1]. The discs contributing to the Floer-theoretic differential come in pairs (u_1, u_2), with u_1 and u_2 elements of the same moduli space. These come from discs in Σ itself, of equal α-areas and disjoint from the diagonal $\delta \subset \operatorname{Sym}^g(\Sigma)$. Thus they have equal ω_0-area when ω_0 is a form as in Proposition 1.1 representing η. When $\lambda = 0$ the lemma therefore follows from Ozsváth–Szabó's calculation.

Now perturb ω_0 to ω_λ with $[\omega_\lambda] = \eta + \lambda\theta$ by adding a small multiple of a closed $(1,1)$-form supported near δ and representing the dual of $[\delta]$. This perturbation does not change the areas, and so u_1 still cancels u_2 and the argument goes through. □

The Floer homology of (T_0, T_1) is usually smaller:

Lemma 5.2. *Either* $\mathrm{rank}_{\Lambda_{\mathbb{Z}/2}} HF_*(T_0, T_1) = 2^{g-2}$, *or* $\int_D \alpha = (1+\lambda)\int_A \alpha - 2\lambda$.

Proof. Again, the proof rests on a calculation from [9] (Lemma 9.4). We set things up as in [9, Figure 9], intersecting $\gamma_0 \times \gamma_2$ with $\gamma_1 \times \gamma_2'$, where γ_2' is a small, transverse, Hamiltonian perturbation of γ_2 such that $\gamma_2 \cap \gamma_2' = 2$. Let $\gamma_2 \cap \gamma_2' = \{u, v\}$ and $\gamma_0 \cap \gamma_1 = \{x, y\}$. Floer's complex is then a tensor product $C' \otimes_{\Lambda_{\mathbb{Z}/2}} C''$, where C'' (the 'uninteresting' part) corresponds to intersections among the curves $(\gamma_3, \ldots, \gamma_g)$ and their α-Hamiltonian translates, while C' has four generators: $\mathbf{x} = u{+}x$, $\mathbf{y} = u{+}y$, $\mathbf{x}' = v{+}x$ and $\mathbf{y}' = v{+}y$. We set up the labelling so that the only potentially non-zero matrix entries in the differential ∂ on C' are $\langle \partial\mathbf{x}, \mathbf{y}\rangle$ and $\langle \partial\mathbf{x}', \mathbf{y}'\rangle$. Now, writing elements of $\Lambda_{\mathbb{Z}/2}$ as sums $\sum_{r\in\mathbb{R}} n_r t^r$ with $n_r \in \mathbb{Z}/2$, Ozsváth–Szabó's calculation shows that

$$\langle \partial\mathbf{x}, \mathbf{y}\rangle = t^{\int_{D_1}\omega_\lambda} + t^{\int_{D_2}\omega_\lambda}$$

where D_1 and D_2 are two particular discs and the integrals are with respect to ω_λ. The first disc, D_1, is the product of the disc D shown in Figure 1 and a constant disc at u. It does not intersect the diagonal δ, and so (using perturbed Kähler forms ω_λ as in the proof of the last lemma) its area is $\int_D \alpha$, regardless of λ. The second disc, D_2, arises from a branched double covering of a disc by the annulus A. By Riemann–Hurwitz, its intersection number with δ is 2. Hence, using the formula $[\delta] = \eta - \theta$ (which was mentioned following Proposition 1.1), we have

$$\int_{D_2} \omega_\lambda = \int_A \alpha + \lambda\left(\int_A \alpha - 2\right).$$

The homology of C' is zero unless the areas of D_1 and D_2 are equal, that is, unless $\int_D \alpha = (1+\lambda)\int_A \alpha - 2\lambda$. But $H_*(C'')$ has rank 2^{g-2} (as in the proof of the previous lemma), so the result follows. □

The 'only if' clause in statement (2) of Theorem 1.2 follows immediately. The 'if' clause also follows, by the argument explained in the last paragraph of Section 4.

6. Application to Heegaard Floer homology

We now turn to the proof of Corollary 1.3, freely drawing on the language of Heegaard Floer homology developed by Ozsváth–Szabó in [9]. Before we begin the proof, we make some remarks on the relation of Heegaard Floer theory to Lagrangian Floer homology.

It is a feature of Lagrangian Floer homology that the moduli spaces defining the differential depend on the Lagrangians and the almost complex structure but do not directly involve a symplectic form. The primary role of symplectic forms is to fix the area of the pseudo-holomorphic 'Whitney discs' in a given homotopy class. Having an *a priori* bound on area is essential for compactness, and for this reason it is important to work with Lagrangian (not merely totally real) submanifolds. Lacking a convenient symplectic form, Ozsváth and Szabó bound the areas of Whitney discs by treating $\mathrm{Sym}^g(\Sigma)$ as a 'symplectic orbifold'—the quotient of $\Sigma^{\times g}$, with its product symplectic form, by the action of the symmetric group—and estimating areas 'upstairs'.

The symplectic class plays a second role in Floer theory: the areas of the holomorphic discs are typically recorded in a Novikov ring of coefficients, just as in Section 5. This is useful because there may be infinitely many homotopy classes of Whitney discs, but there are only finitely many index 1 Whitney discs with a given area.

Ozsváth–Szabó are able to do without Novikov rings by showing that, under the 'weak admissibility' assumption on Heegaard diagrams, there are only finitely many homotopy classes of Whitney discs with a given Maslov index, fixed intersection number with the divisor $z + \mathrm{Sym}^{g-1}(\Sigma)$, and additionally satisfying a positivity constraint automatically satisfied by pseudo-holomorphic discs [9, Lemma 4.14].

Now suppose ω_λ is as in Proposition 1.1. Lagrangian Floer homology for $(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ (in any of its possible algebraic variants) can be computed via any almost complex structure taming ω_λ that satisfies a regularity condition. Ozsváth and Szabó show that to attain regularity it is sufficient to consider (paths in) a particular class of almost complex structures, those that are small 'nearly symmetric' perturbations of the standard integrable structure [9, Theorem 3.4] (these still tame ω_λ). The periods are controlled by the Maslov index and intersection numbers with $z + \mathrm{Sym}^{g-1}(\Sigma)$, and the groups are set up so as to keep track of these intersection numbers.

Heegaard tori are monotone Lagrangians of minimal Maslov index 2. The troublesome feature of Lagrangian Floer theory in such a case is that bubbling-off of discs spoils compactness of the moduli spaces involved in the definition of the groups (more precisely, the matrix coefficients $\langle \partial \circ \partial x, x\rangle$ where x is a generator for the Floer complex and ∂ the usual Floer-theoretic 'differential'), cf. [8]. Ozsváth and Szabó show that, though Maslov index 2 discs can bubble off in the relevant 1-dimensional moduli space, they always do so in cancelling pairs [9, Theorem 3.15], which, as explained in [8], implies that $\partial \circ \partial = 0$.

Proof of Corollary 1.3. We shall set up a 'continuation isomorphism'

$$HF^+(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}) \to HF^+(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}),$$

and a similar one for the other variants of Heegaard Floer homology.

By hypothesis $\mathbb{T}_\alpha$ is transverse to $\mathbb{T}_\beta$, and perturbing the $\boldsymbol{\alpha}'$ curves slightly we may assume that $\mathbb{T}_\alpha$ is transverse to $\mathbb{T}_\beta$. Choose any small $\lambda > 0$ and an area form α satisfying the area constraint of Theorem 1.2 (2). We then consider a symplectic form ω_λ as in Proposition 1.1 and use Theorem 1.2 to obtain a Hamiltonian isotopy $\{\phi_t\}_{t\in[0,1]}$ with $\phi_0 = \mathrm{id}$ and $\phi_1(\mathbb{T}_\alpha) = \mathbb{T}_{\alpha'}$.

By a general principle in Lagrangian Floer homology, the Hamiltonian isotopy $\{\phi_t\}_{t\in[0,1]}$ induces a continuation isomorphism $HF_*(\mathbb{T}_\alpha, \mathbb{T}_\beta; \omega_\lambda) \to HF_*(\phi_1(\mathbb{T}_\alpha), \mathbb{T}_\beta; \omega_\lambda)$ (Floer homology with Novikov-ring coefficients). Because the minimal Maslov index is 2, the continuation-map moduli spaces cannot bubble here (and the same goes for those used in proving the composition law for continuation maps). However, it is not immediately apparent that the continuation map makes sense for HF^+ (say), because the ω_λ-area of Whitney discs might not be controlled by Maslov index and intersection number with $z + \mathrm{Sym}^{g-1}(\Sigma)$. We deal with this issue now.

The continuation map can be understood as follows. We have a Hamiltonian isotopy $\{\phi_s\}_{s\in[0,1]}$, and we may extend this to a family $\{\phi_s\}_{s\in\mathbb{R}}$ where $\phi_s = \mathrm{id}$ for $s<0$ and $\phi_s = \phi_1$ for $s>1$ (this family will not be smooth, so strictly we should reparametrise $[0,1]$ before we extend so as to obtain a smooth family). We consider the trivial $\mathrm{Sym}^g(\Sigma)$-bundle $\mathrm{Sym}^g(\Sigma)\times\mathbb{R}\times[0,1]\to\mathbb{R}\times[0,1]$, and make it into a Hamiltonian fibration by giving endowing it with a closed 2-form $\Omega := \omega_\lambda + d(\beta(t)H_s ds)$. Here s is the $\mathbb{R}$-coordinate, t the $[0,1]$-coordinate; $\{H_s\}_{s\in\mathbb{R}}$ is a family of functions, non-zero only for $s\in[0,1]$, generating ϕ_s; and $\beta(t)$ is a cut-off function, equal to 1 near $t=1$ and to 0 near $t=0$. There are Lagrangian boundary conditions $\mathbb{T}_\beta\times\mathbb{R}\times\{1\}$ and $\bigcup_{s\in\mathbb{R}}\phi_s(\mathbb{T}_\alpha)\times\{s\}\times\{0\}$ over the boundary of the strip. The continuation map is defined by counting index-0 pseudo-holomorphic sections of this fibration subject to the Lagrangian boundary conditions (for an almost complex structure making the projection holomorphic, and translation-invariant for $|s|\gg 0$). See [15] for details of the fibre-bundle approach to Floer theory.

We need to estimate the energies of sections u subject to the Lagrangian boundary conditions and asymptotic to intersection points $\mathbf{x}_-\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ as $s\to-\infty$ and $\mathbf{x}_+\in\phi_1(\mathbb{T}_\alpha)\cap\mathbb{T}_\beta$ as $s\to+\infty$.

These energies are cohomological in nature; they are invariant under compactly supported homotopies of u, for instance. Because of this, we can estimate the energies by considering a degeneration of the domain in which a disc is pinched off, as in Figure 2. That is, we consider a 1-parameter family of surfaces $\{S_t\}_{t\in[0,\infty)}$, with $S_0=\mathbb{R}\times[0,1]$, pulling back our Hamiltonian fibration and Lagrangian boundary conditions by a family of diffeomorphisms $f_t\colon S_t\to S_0$. We arrange that the surfaces S_t limit to the nodal surface S_∞ shown in Figure 2 as $t\to\infty$. Conformally, we can view S_∞ as the nodal union of a triangle T and a disc D. We can arrange that there is a limiting Hamiltonian fibration over S_∞ (still smoothly trivialised), and that over the three edges of the triangle, we have constant Lagrangian boundary conditions $\mathbb{T}_\alpha$, $\phi_1(\mathbb{T}_\alpha)$ and $\mathbb{T}_\beta$, as shown. Over the boundary of the disc we have the Lagrangian boundary condition $\phi_s(\mathbb{T}_\alpha)$, where s now parametrises the boundary anticlockwise.

Consider triangles $T\to\mathrm{Sym}^g(\Sigma)$, subject to the boundary conditions given by $\mathbb{T}_\alpha$, $\mathbb{T}_{\alpha'}=\phi_1(\mathbb{T}_\alpha)$ and $\mathbb{T}_\beta$, of Maslov index 0, and with fixed intersection number with $z+\mathrm{Sym}^{g-1}(\Sigma)$. Assume that T is asymptotic to intersection points $\mathbf{x}_\pm$, as above, whose associated Spin^c-structure is $\mathfrak{s}$. According to [9, Section 9.2], the number of finite-energy holomorphic triangles T satisfying these conditions is finite. On the other hand, homotopy classes of sections over D, subject to the boundary conditions $\phi_s(\mathbb{T}_\alpha)$ and mapping the node to a fixed $\mathbf{x}_0\in\mathbb{T}_\alpha\cap\phi_1(\mathbb{T}_\alpha)$, correspond to $\pi_2(\mathrm{Sym}^g(\Sigma),\mathbb{T}_\alpha)=\mathbb{Z}$ (by extending them to sections of a trivial fibration over a larger disc, with constant Lagrangian boundary condition $\mathbb{T}_\alpha$), and so again the energy is controlled by the intersection number with $z+\mathrm{Sym}^{g-1}(\Sigma)$.

Each homotopy class of sections over S_∞ can be smoothed out to a homotopy class of sections of S_t for t finite, and it is easy to see that every homotopy class of sections over S_0 arises this way. The smoothing leaves the energy unchanged. Hence, going

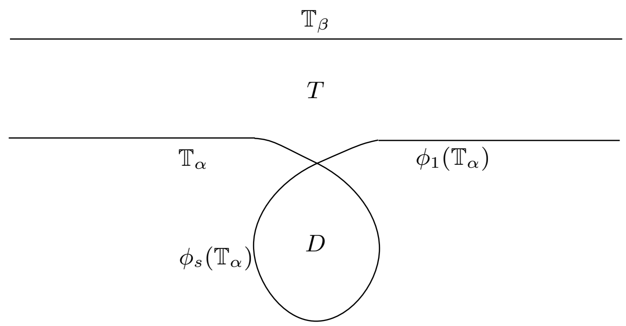

FIGURE 2. Degeneration of the domain to the union of a (conformal) triangle and a disc.

back to the continuation map picture (over S_0), we conclude that the energy of index-0 pseudo-holomorphic strips with fixed asymptotics and fixed intersection number with $z + \mathrm{Sym}^{g-1}(\Sigma)$ is bounded. Thus the continuation map is well-defined. More accurately, the continuation map is defined initially on CF^∞, but it respects the subcomplexes CF^- and so induces maps on CF^- and CF^+. For $\widehat{CF}$ one defines the continuation map using strips whose intersection number with $z + \mathrm{Sym}^{g-1}(\Sigma)$ is zero.

By using the same degeneration to obtain the necessary energy bounds, one sees that these continuation maps are chain maps, and that the continuation map associated with the reverse isotopy ϕ_{1-t} is an inverse up to chain-homotopy. Hence all the continuation maps are quasi-isomorphisms.

Note finally that, because Theorem 1.2 produces Hamiltonian isotopies which are canonical (up to homotopy with fixed endpoints), the continuation isomorphisms on Heegaard Floer homology are also canonical. $\square$

Remark 6.1. The continuation isomorphisms on Heegaard Floer homology which we have associated with a handleslide are the same as the isomorphisms constructed by Ozsváth–Szabó using holomorphic triangles. To see this one applies a gluing theorem for holomorphic sections to the degeneration of Figure 2.

7. Branched coverings and smoothing of currents

This final section reproduces the note [10], which has not previously been published.

We consider branched coverings $\pi\colon X \to X'$ of complex manifolds—that is, holomorphic maps which are proper, surjective, and finite. The branch locus $B_\pi \subset X'$ of such a map is

$$B_\pi = \{\pi(x) : x \in X,\ \ker D_x(\pi) \neq 0\}.$$

A C^∞ Kähler form ω on X can be pushed forward—in the sense of currents, that is, of 2-forms with coefficients of class L^1_{loc}—to a closed current $\pi_*(\omega)$ on X' which is smooth on $X' \setminus B_\pi$. The following result is essentially due to Varouchas [16].

Proposition 7.1. *Let $\pi\colon X \to X'$ be a branched covering of complex manifolds, and ω a Kähler form on X. Let N be a neighbourhood of the branch locus in X'. Then there exists a Kähler form ω' on X', representing the class $[\omega'] = \pi_*[\omega] \in H^2(X';\mathbb{R})$, such that $(\pi_*\omega - \omega')|_{X'\setminus \overline{N}} = 0$.*

We explain here the minor modification of Varouchas' argument from [16] needed to prove this result (the stated conclusion in [16] is simply that X' admits a Kähler form).

Proposition 7.1 has the following special case:

Corollary 7.2. *Let Σ be a Riemann surface with positive area form α. Let $\pi\colon \Sigma^{\times n} \to \mathrm{Sym}^n(\Sigma)$ be the projection map. Suppose that $N \subset \mathrm{Sym}^n(\Sigma)$ is an open subset containing the diagonal δ. Then there exists a Kähler form ω on $\mathrm{Sym}^n(\Sigma)$, representing the class $\eta = \pi_*[\alpha^{\times n}]$, such that outside $\overline{N}$, ω is the smooth push-forward $\pi_*(\alpha^{\times n})$ of the product form.*

Definition 7.1. Let X be a complex manifold. A **Kähler cocycle** on X is a collection $(U_i, \varphi_i)_{i\in I}$, where $(U_i)_{i\in I}$ is an open cover of X, and $\varphi_i\colon U_i \to \mathbb{R}$ is a a function, such that for all $i, j \in I$,

1. φ_i is strictly plurisubharmonic on U_i; and
2. $\varphi_i - \varphi_j$ is pluriharmonic on $U_i \cap U_j$.

One ascribes to the cocycle a property (continuity, smoothness, etc.) possessed by all the φ_i. Kähler cocycles are, by definition, upper semicontinuous. Condition (1) means that the 2-current $dd^c\varphi_i$ is strictly positive on U_i; (2) means that these currents agree on overlaps, and are therefore restrictions of a 2-current ω on X (closed and strictly positive). If the cocycle is C^∞ then ω will be a Kähler form.

Varouchas' *lemme principal* is the following. The proof uses the 'regularised maximum' technique of Richberg and Demailly.

Lemma 7.3. *Let U, V, W, Ω be open subsets of $\mathbb{C}^n$ with*

$$U \Subset V \Subset W, \quad \Omega \subset W.$$

Let $\phi\colon W \to \mathbb{R}$ be continuous, strictly plurisubharmonic, and smooth on Ω. Then there exists a function $\psi\colon W \to \mathbb{R}$, again continuous and strictly plurisubharmonic, equal to ϕ on $W \setminus \overline{V}$ and smooth on $U \cup \Omega$.

(The notation $U \Subset V$ means that $\overline{U} \subset V$.) One passes from local to global by the following argument, which we give in detail since Varouchas' stated conclusion is weaker.

Lemma 7.4. *Let $(U_i, \varphi_i)_{i \in I}$ be a continuous Kähler cocycle on the complex manifold X. Suppose that $X = X_1 \cup X_2$, with X_1 and X_2 open, and that the functions $\varphi_i|_{U_i \cap X_1}$ are smooth. Then there exists a continuous function $\chi\colon X \to \mathbb{R}$ supported in X_2 and a locally finite refinement $V_j \subset U_{i(j)}$ $\quad (j \in J)$ so that the family*

$$\big(V_j, \varphi_{i(j)}|_{V_j} + \chi|_{V_j}\big)_{j \in J}$$

is a smooth Kähler cocycle.

Proof. Refine the cover $(U_i)_{i\in I}$ to a countable, locally finite cover $(V_i)_{i \in I_1 \amalg I_2}$ with the property that

$$i \in I_\alpha \;\Rightarrow\; V_i \subset X_\alpha, \quad \alpha = 1, 2.$$

For definiteness suppose both I_1 and I_2 are infinite; say $I_\alpha = \mathbb{N} \times \{\alpha\}$, so that the labels i are pairs $(k, 1)$ with $k \in \mathbb{Z}$ for I_1, or $(k, 2)$ for I_2. Find open subsets $V_i'' \Subset V_i' \Subset V_i$ such that (V_i'') still covers X. Set

$$A_1 = \emptyset, \quad A_n = V''_{(1,2)} \cup \cdots \cup V''_{(n-1,2)} \quad (n > 1).$$

The sets A_n exhaust $X_2 \setminus X_1$. Let $(V_i, \psi_i^1)_{i \in I_1 \cup I_2}$ be the Kähler cocycle induced from $(U_i, \varphi_i)_{i \in I}$ by the refinement.

Claim: there are Kähler cocycles (V_i, ψ_i^n), where $n = 1, 2, \ldots$ indexes the elements of I_2, such that the following hold for all $i \in I_1 \cup I_2$ and all $n > 1$:

1. ψ_i^n is smooth on the set $V_i \cap (X_1 \cup A_n)$.
2. There is a continuous function $\chi_n\colon X \to \mathbb{R}$, with $\mathrm{Supp}(\chi_n) \subset V'_{(n-1,2)}$, such that $\psi_i^n = \psi_i^{n-1} + \chi_n$.

We prove the claim by induction on n. Apply the 'lemme principal' to

$$(U, V, W, \Omega) = (V''_{(n-1,2)}, V'_{(n-1,2)}, V_{(n-1,2)}, V_{(n-1,2)} \cap (X_1 \cup A_{n-1}))$$

and to the function $\psi^{n-1}_{(n-1,2)}$, obtaining a new function $\psi^n_{(n-1,2)}$; let $\chi_n = \psi^n_{(n-1,2)} - \psi^{n-1}_{(n-1,2)}$, extended by zero to all of X, and use (2) to define the new cocycle. We have to verify (1), i.e. to prove smoothness of ψ_i^n at each $x \in V_i \cap (X_1 \cup A_n)$. If $x \notin V'_{(n-1,2)}$ then $\chi_n(x) = 0$, but ψ_i^{n-1} was already smooth. If $x \in V'_{(n-1,2)}$ then, near x, $\psi_i^n = (\psi_i^n - \psi^n_{(n-1,2)}) + \psi^n_{(n-1,2)} = (\psi_i^1 - \psi^1_{(n-1,2)}) + \psi^n_{(n-1,2)}$, which is the sum of a pluriharmonic function and a smooth plurisubharmonic one. But a pluriharmonic function is automatically smooth. By a similar argument, ψ_i^n is strictly plurisubharmonic.

Now define a function $\chi\colon X \to \mathbb{R}$ by $\chi(x) = \sum_{n \geq 1} \chi_n(x)$ (the sum is locally finite). Then $\psi_i^\infty(x) := \psi_i^1(x) + \chi(x)$ defines a Kähler cocycle. It is smooth, since on one hand, on $X \setminus \bigcup V'_{(n,2)} \subset X_1$, the original cocycle was smooth and has not been modified, while on the other hand,

$$V_{(n,2)} \subset X_1 \cup \bigcup A_k,$$

so smoothness on $V_{(n,2)}$ is guaranteed by (1). Hence χ has the required properties. $\square$

Proof of Proposition 7.1. Each fibre $\pi^{-1}(x')$, being finite, has a neighbourhood which is a disjoint union of open balls. Hence, using the dd^c-lemma, one can find a smooth Kähler cocycle (U_i, φ_i) on X such that each U_i contains a fibre of π, with $\omega|_{U_i} = dd^c\varphi_i$. One can then find a locally finite cover (U_i') of X' such that $U_i \supset \pi^{-1}(U_i')$.

A general property of branched covers is that the push-forward $\pi_* f$ of a continuous function $f\colon X \to \mathbb{R}$ is again continuous (it is given by $\pi_* f(x') = \sum_{x\in\pi^{-1}(x')} f(x)$, where the points x are taken with multiplicities). The family $(U_i', \pi_*\varphi_i)$ on X' is thus a continuous Kähler cocycle: plurisubharmonicity is clear away from B_π, since $\pi_* dd^c f = dd^c \pi_* f$, hence everywhere by density; similarly for pluriharmonicity on overlaps. As for strictness, let $\omega_{\mathbb{C}^n}$ be the standard Kähler form in a complex chart centred at a point $x \in B_\pi$. Then, for a test $(2n-2)$-form β supported near x, one has

$$(\pi_* dd^c f - \epsilon\omega)(\beta) = \pi_*(dd^c f - (\deg\pi)^{-1}\epsilon\pi^*\omega_{\mathbb{C}^n})(\eta) = (dd^c f - (\deg\pi)^{-1}\epsilon\pi^*\omega_{\mathbb{C}^n})(\pi^*\eta),$$

and using this one verifies strict positivity of $\pi_* dd^c f$.

Now let $N' \Subset N$ be a smaller open neighbourhood of B_π. Apply the 'global smoothing' lemma (7.4) to $(U_i', \pi_*\varphi_i)$ on X', taking $X_1 = X' \setminus \overline{N'}$ and $X_2 = N$. The output is a function $\chi\colon X' \to \mathbb{R}$, as well as a refinement of (U_i'), such that

$$\omega_{X'} := dd^c(\pi_*\varphi_i + \chi) = \pi_* dd^c\varphi_i + dd^c\chi$$

is a well-defined 2-form with the right properties. For any smooth, closed test form $\beta \in \Omega_c^{2n-2}(X')$ one has $\int_{X'} dd^c\chi \wedge \beta = 0$, hence $dd^c\chi$ represents the zero cohomology class. $\square$

References

[1] Arbarello, E., Cornalba, M., Griffiths, P and Harris, J., *Geometry of algebraic curves*, volume 1, Springer-Verlag, 1985.

[2] Bertram, A. and Thaddeus, M., *On the quantum cohomology of a symmetric product of an algebraic curve*, Duke Math. J. 108 (2001), no. 2, 329–362.

[3] Donaldson, S. and Smith, I., *Lefschetz pencils and the canonical class for symplectic four-manifolds.* Topology 42 (2003), no. 4, 743–785.

[4] Gompf, R., *A new construction of symplectic manifolds*, Ann. of Math. (2) 142 (1995), no. 3, 527–595.

[5] Gompf, R. and Stipsicz, A., *Four-manifolds and Kirby calculus*, Graduate Studies in Mathematics, 20. American Mathematical Society, Providence, RI, 1999

[6] Ionel, E.-N. and Parker, T., *The symplectic sum formula for Gromov–Witten invariants*, Ann. of Math. (2) 159 (2004), no. 3, 935–1025.

[7] McDuff, D., and Salamon, D., *Introduction to symplectic topology*, second edition, Oxford University Press, New York, 1998.

[8] Oh, Y.-G., *Addendum to: "Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I"*, Comm. Pure Appl. Math. 48 (1995), no. 11, 1299–1302.

[9] Ozsváth, P., and Szabó, Z., *Holomorphic disks and topological invariants for closed three–manifolds*, Ann. of Math. (2) 159 (2004), no. 3, 1027–1158.

[10] Perutz, T., *A remark on Kähler forms on symmetric products of Riemann surfaces*, 2005 preprint, ArXiv: math/0501547.

[11] Perutz, T., *Lagrangian matching invariants for fibred four-manifolds: I*, Geom. Topol. 11, 2007, 759–828.

[12] Perutz, T., *Symplectic fibrations and the abelian vortex equations*, Commun. Math. Phys., 278 (2008), no. 2, 289–306.

[13] Ran, Z., *A note on Hilbert schemes of nodal curves*, J. Algebra 292 (2005), no. 2, 429–446.

[14] Seidel, P., *Lectures on four-dimensional Dehn twists*, 2003 preprint, ArXiv: math/0309012v3.

[15] Seidel,. P., *A long exact sequence for symplectic Floer cohomology*, Topology, 42 (2003), no. 5, 1003–1063.

[16] Varouchas, J., *Stabilité de la classe des variétés Kählériennes par certains morphismes propres,* Invent. Math. 77 (1984), 117–127.

[17] Voisin, C., *Hodge theory and complex algebraic geometry, I*, transl. L. Schneps. Cambridge Studies in Advanced Mathematics, 76, Cambridge University Press, 2002.

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Construction of symplectic cohomology $S^2 \times S^2$

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Abstract. In this article, we present symplectic 4-manifolds with the same integral cohomology as $S^2 \times S^2$. A generalization of this construction is given as well, an infinite family of symplectic 4-manifolds cohomology equivalent to $\#_{(2g-1)}(S^2 \times S^2)$ for any $g \geq 2$. We also compute the Seiberg-Witten invariants of the 4-manifolds we construct.

1. Introduction

The aim of this article is to construct examples of symplectic 4-manifolds with the same integral cohomology as $S^2 \times S^2$. Similar problems have been studied in the algebro-geometric category, i.e. existence of algebraic surfaces homology equivalent but not isomorphic to $\mathbf{P^2}$ (or $\mathbf{P^1} \times \mathbf{P^1}$) as an algebraic variety. D. Mumford [Mu] and R. Pardini [P] gave the constructions of such fake $\mathbf{P^2}$ and $\mathbf{P^1} \times \mathbf{P^1}$.

We study this problem in the symplectic category. Our main results are the following two theorems.

Theorem 1.1. *Let K be a genus one fibered knot in S^3. Then there exist a minimal symplectic 4-manifold X_K cohomology equivalent to $S^2 \times S^2$.*

Theorem 1.2. *Let K be a genus one and K' be any genus $g \geq 2$ fibered knot in S^3. Then there exist an infinite family of minimal symplectic 4-manifolds $V_{KK'}$ that is cohomology equivalent to $\#_{(2g-1)}(S^2 \times S^2)$.*

This paper is organized as follows: Section 2 contains the basic definitions and formulas that will be important throughout this paper. Section 3 gives a quick introduction to Seiberg-Witten invariants. The remaining two sections are devoted to the construction of family of symplectic 4-manifolds cohomology equivalent to $\#_{(2g-1)}(S^2 \times S^2)$ and the fundamental group computation for our examples.

2. Preliminaries

2.1. Fibered Knots

In this section, we give a short introduction to fibered knots and state a few facts that will be needed in our construction. We refer the reader to Section 10.H [R] for a more complete treatment.

Definition 2.1. Let p and q be relatively prime positive integers. The knot which wraps around the solid torus in the longitudinal direction p times and in the meridianal direction q times is called the (p,q) *torus knot* and denoted as $T_{p,q}$.

Lemma 2.1. [S] *a) The group of the torus knot $T_{p,q}$ can be represented as follows:*

$$\pi_1(S^3 \setminus T_{p,q}) \quad = \quad \langle u, v \mid u^p = v^q \rangle$$

b) The elements $m = u^s v^r$, $l = u^p m^{-pq}$, where $pr + qs = 1$, describe meridian and longitude of the $T_{p,q}$ for a suitable chosen basepoint.

All torus knots belong to the larger category of fibered knots.

Definition 2.2. A knot K in S^3 is called *fibered* if there is fibration $f : S^3 \setminus K \longrightarrow S^1$ behaving "nicely" near K. This means that K has a neighbourhood framed as $S^1 \times D^2$, with $K \cong S^1 \times 0$ and restriction of the map f to $S^1 \times (D^2 - 0)$ is the map to S^1 given by $(t, x) \to x/|x|$.

It follows from the definition that a preimage for each point $p \in S^1$ is the Seifert surface for the given knot. The genus of this Seifert surface will be called the genus of the given fibered knot.

The fibered knots form a large class among the all classical knots. Below we state two theorems that can be used to detect if the given knot is fibered or not.

Theorem 2.2. [S] *The knot $K \subset S^3$ is a fibered knot of genus g if and only if the commutator subgroup of its knot group $\pi_1(S^3 \backslash K)$ is finitely-generated and free group of rank $2g$.*

Theorem 2.3. [BZ] *The Alexander polynomial $\Delta_K(t)$ of a fibered knot in S^3 is monic, i.e. the first and the last non-zero coefficients of $\Delta_K(t)$ are ± 1.*

If a genus one knot is fibered, then it can be shown by above theorems and also by explicit construction [BZ] that it must be either the trefoil or the figure eight knot. Also, one can construct infinitely many fibered knots for a fixed genus $g \geq 2$.

2.2. Generalized fiber sum

Definition 2.3. Let X and Y be closed, oriented, smooth 4-manifolds each containing a smoothly embedded surface Σ of genus $g \geq 1$. Assume Σ represents a homology class of infinite order and has self-intersection zero in X and Y, so that there exists a product tubular neighborhood, say $\nu\Sigma \cong \Sigma \times D^2$, in both X and Y. Using an orientation-reversing and fiber-preserving diffeomorphism $\psi : \Sigma \times S^1 \to \Sigma \times S^1$, we can glue $X \setminus \nu\Sigma$ and $Y \setminus \nu\Sigma$ along the boundary $\partial(\nu\Sigma) \cong \Sigma \times S^1$. The resulting closed oriented smooth 4-manifold, denoted by $X \#_\psi Y$, is called a *generalized fiber sum* of X and Y along Σ, determined by ψ.

Definition 2.4. Let $e(X)$ and $\sigma(X)$ denote the Euler characteristic and the signature of a closed oriented smooth 4-manifold X, respectively. We define

$$c_1^2(X) := 2e(X) + 3\sigma(X), \quad \chi_h(X) := \frac{e(X) + \sigma(X)}{4}.$$

Lemma 2.4. *Let X and Y be closed, oriented, smooth 4-manifolds containing an embedded surface Σ of self-intersection 0. Then*

$$\begin{aligned} c_1^2(X \#_\psi Y) &= c_1^2(X) + c_1^2(Y) + 8(g-1), \\ \chi_h(X \#_\psi Y) &= \chi_h(X) + \chi_h(Y) + (g-1), \end{aligned}$$

where g is the genus of the surface Σ.

Proof. The above simply follow from the well-known formulas

$$e(X \#_\psi Y) = e(X) + e(Y) - 2e(\Sigma), \quad \sigma(X \#_\psi Y) = \sigma(X) + \sigma(Y). \qquad \square$$

If X and Y are symplectic 4-manifolds and Σ is a symplectic submanifold in both, then according to a theorem of Gompf [Go], $X \#_\psi Y$ admits a symplectic structure. In such a case, we will call $X \#_\psi Y$ a *symplectic sum*.

To show the minimality of our symplectic manifolds, we use the following theorem of M. Usher [U]. In order to state his theorem, we slightly abuse the notation for the fiber sum above.

Theorem 2.5. [U] **(Minimality of Symplectic Sums)** *Let $Z = X_1 \#_{F_1 = F_2} X_2$ be sympletic fiber sum of manifolds X_1 and X_2. Then:*

(i) If either $X_1 \backslash F_1$ or $X_2 \backslash F_2$ contains an embedded sympletic sphere of square -1, then Z is not minimal.

(ii) If one of the summands X_i (say X_1) admits the structure of an S^2-bundle over a surface of genus g such that F_i is a section of this fiber bundle, then Z is minimal if and only if X_2 is minimal.

(iii) In all other cases, Z is minimal.

3. Seiberg-Witten Invariants

In this section, we review the basics of Seiberg-Witten invariants introduced by Seiberg and Witten. Let us recall that the Seiberg-Witten invariant of a smooth closed oriented

4-manifold X with $b_2^+(X) > 1$ is an integer valued function which is defined on the set of *spin*^{c} structures over X [W]. For simplicity, we assume that $H_1(X, \mathbb{Z})$ has no 2-torsion. Then there is a natural identification of the *spin*^{c} structures of X with the characteristic elements of $H^2(X, \mathbb{Z})$ as follows: to each *spin*^{c} structure $\mathfrak{s}$ over X corresponds a bundle of positive spinors $W_{\mathfrak{s}}^+$ over X. Let $c(\mathfrak{s}) \in H_2(X)$ denote the Poincaré dual of $c_1(W_{\mathfrak{s}}^+)$. Each $c(\mathfrak{s})$ is a characteristic element of $H_2(X, \mathbb{Z})$ (i.e. its Poincaré dual $\hat{c}(\mathfrak{s}) = c_1(W_{\mathfrak{s}}^+)$ reduces mod 2 to $w_2(X)$).

In this set up, we can view the Seiberg-Witten invariant as an integer valued function

$$\mathrm{SW}_X : \{K \in H^2(X, \mathbb{Z}) | K \equiv w_2(TX) \pmod 2\} \to \mathbb{Z}.$$

If $\mathrm{SW}_X(\beta) \neq 0$, then we call β a *basic class* of X. It is a fundamental fact that the set of basic classes is finite. Furthermore, if β is a basic class, then so is $-\beta$ with

$$\mathrm{SW}_X(-\beta) = (-1)^{(e+\sigma)(X)/4}\,\mathrm{SW}_X(\beta)$$

where $e(X)$ is the Euler characteristic and $\sigma(X)$ is the signature of X.

When $b_2^+(X) > 1$, then Seiberg-Witten invariant is a diffeomorphism invariant. It does not depend on the choice of generic metric on X or a generic perturbation of Seiberg-Witten equations.

If $b_2^+(X) = 1$, then the Seiberg-Witten invariant depends on the chosen metric and perturbation of Seiberg-Witten equations. Let us recall that the perturbed Seiberg-Witten moduli space $\mathcal{M}_X(\beta, g, h)$ is defined as the solutions of the Seiberg-Witten equations

$$F_A^+ = q(\psi) + ih, \; D_A\psi = 0$$

divided by the action of the gauge group, where A is a connection on the line bundle L with $c_1(L) = \beta$, g is Riemannian metric on X, ψ is the section of the positive spin bundle corresponding to the *spin*^{c} structure determined by β, F_A^+ is a self-dual part of the curvature F_A, D_A is the twisted Dirac operator, q is a quadratic function, and h is self-dual 2-form on X. If $b_2^+(X) \geq 1$ and h is generic metric, then Seiberg-Witten moduli space $\mathcal{M}_X(\beta, g, h)$ is a closed manifold with dimension $d = (\beta^2 - 2e(X) - 3\sigma(X))/4$. The Seiberg-Witten invariant is defined as follows:

$$\begin{cases} \mathrm{SW}_X(\beta) = \langle [\mathcal{M}_X(\beta, g, h)], \mu^{d/2} \rangle & \text{if } d \geq 0 \text{ and even} \\ \mathrm{SW}_X(\beta) = 0 & \text{otherwise} \end{cases}$$

where $\mu \in H^2(\mathcal{M}_X(\beta, g, h), \mathbb{Z})$ is the Euler class of the base fibration.

When $b_2^+(X) = 1$, the Seiberg-Witten invariant $SW_X(\beta, g, h)$ depends on g and h. Because of this, there are two types of Seiberg-Witten invariants: SW_X^+ and SW_X^-.

Theorem 3.1. [KM], [OS] *Let X be closed, oriented, smooth 4-manifold with $b_1(X) = 0$ and $b_2^+(X) = 1$. Fix a homology orientation of $H_+^2(X, \mathbb{R})$. For given Riemannian metric g let ω_+^g be the unique g-harmonic self-dual 2-form that has norm 1 and is compatible with the orientation of $H_+^2(X, \mathbb{R})$. Then for each characteristic element β with $(\beta^2 - 2e(X) - 3\sigma(X))/4 \geq 0$ the following holds: If $(2\pi\beta + h_1) \cdot \omega_+^{g_1}$ and $(2\pi\beta + h_2) \cdot \omega_+^{g_2}$ are not zero and have same sign, then $SW_X(\beta, g_1, h_1) = SW_X(\beta, g_2, h_2)$.*

Definition 3.1. If $(2\pi\beta + h) \cdot \omega_+^g > 0$, then write $\mathrm{SW}_X^+(\beta)$ for $\mathrm{SW}_X(\beta, g, h)$. If $(2\pi\beta + h) \cdot \omega_+^g < 0$, then write $\mathrm{SW}_X^-(\beta)$ for $\mathrm{SW}_X(\beta, g, h)$

Theorem 3.2. [Sz] *Let X be closed, oriented, smooth 4-manifold with $b_1(X) = 0$ and $b_2^+(X) = 1$ and $b_2^- \leq 9$. Then for each characteristic element β, pair of Riemannian metrics g_1, g_2, and small perturbing 2-forms h_1, h_2, we have $SW_X(\beta, g_1, h_1) = SW_X(\beta, g_2, h_2)$.*

Proof. . Let β be a characteristic element for which $d \geq 0$. Then $2e(X) + 3\sigma(X) = 4 + 5b_2^+ - b_2^- \geq 0$, which as implies $\beta^2 \geq 0$. It follows that $\beta \cdot \omega_+^{g_1}$ and $\beta \cdot \omega_+^{g_2}$ are both non-zero and have same signs. Now using the Theorem 3.1, the result follows.

□

Theorem 3.3. [LL] **(Wall crossing formula)** *Assume that X is a closed, oriented, smooth 4-manifold with $b_1(X) = 0$ and $b_2^+(X) = 1$. Then for each characteristic line bundle L on X such that the formal dimension of the Seiberg-Witten moduli space is non-negative even integer $2m$, then $SW_X^+(L) - SW_X^-(L) = -(-1)^m$.*

Note that when $b_2^- \leq 9$, it follows from the above result that there is well defined Seiberg-Witten invariant which will be denoted as $\mathrm{SW}_X^o(X)$.

Theorem 3.4. [T] *Suppose that X is closed symplectic 4-manifold with $b_2^+(X) > 1$ ($b_2^+(X) = 1$). If K_X is a canonical class of X, then $SW_X(\pm K_X) = \pm 1$ ($SW_X^-(-K_X) = \pm 1$).*

Definition 3.2. The 4-manifold X is of simple type if each basic class β satisfies the equation $\beta^2 = {c_1}^2(X) = 3\sigma(X) + 2e(X)$.

Theorem 3.5. [KM], [OS] **(Generalized adjunction formula for $b_2^+ > 1$)** *Assume that $\Sigma \subset X$ is an embedded, oriented, connected surface of genus $g(\Sigma)$ with self-intersection $|\Sigma|^2 \geq 0$ and represents nontrivial homology class. Then for every Seiberg-Witten basic class β, $2g(\Sigma) - 2 \geq |\Sigma|^2 + |\beta(|\Sigma|)|$. If X is of simple type and $g(\Sigma) > 0$, then the same inequality holds for Σ with arbitrary self-intersection.*

Theorem 3.6. [LL] **(Generalized adjunction formula for $b_2^+ = 1$)** *Let $\Sigma \subset X$ is an embedded, oriented, connected surface of genus $g(\Sigma)$ with self-intersection $|\Sigma|^2 \geq 0$ and $[\Sigma]$ represents nontrivial homology class. Then any characteristic class β with $SW_X{}^o(\beta) \neq 0$ satisfies $2g(\Sigma) - 2 \geq \Sigma^2 + |\beta(|\Sigma|)|$.*

4. Symplectic manifolds cohomology equivalent to $S^2 \times S^2$

To construct our manifolds, we will start with well known symplectic 4-manifolds described below. By applying Gompf's symplectic fiber sum operation along genus one and then genus two surfaces, we will obtain our manifolds X_K.

Let K be a genus one fibered knot (i.e. the trefoil or the figure eight knot) in S^3 and m a meridianal circle to K. Let M_K denote 3-manifold obtained as the result of 0-framed Dehn surgery on K. The manifold M_K has the same integral homology as $S^2 \times S^1$, where the class of m generates $H_1(M_K)$. Since the knot K has genus one and is fibered, it follows that the manifold $M_K \times S^1$ is a torus bundle over a torus which is homology equivalent to $T^2 \times S^2$. Since K is a fibered knot, $M_K \times S^1$ admits a symplectic structure. Note that $m \times S^1 = m \times x = T_m$ is a section of this fibration. Both the torus fiber and the torus section are symplectically embedded and have a self-intersection zero. The first homology of $M_K \times S^1$ is generated by the standard first homology generators m and x of the torus section. The generators γ_1 and γ_2 of the fiber F, coming from Seifert surface of knot K, are trivial in homology.

The intermediate building block in our construction will be a twisted fiber sum of two copies of the manifold $M_K \times S^1$, where we identify a fiber of first fibration with a section of the second one. Let Y_K denote the mentioned twisted fiber sum $Y_K = M_K \times S^1 \#_{F=T_m} M_K \times S^1$. It follows from Gompf's theorem [Go] that Y_K is symplectic. Notice that the manifold Y_K is obtained by knot surgery operation of Fintushel-Stern [FS1] from manifold $M_K \times S^1$. Thus $Y_K = (M_K \times S^1)_K$. In this step we could have also chosen a different genus 1 fibered knot.

Let T_1 be the section of the first copy of $M_K \times S^1$ and T_2 be the fiber of the second copy. Note that $T_1 \# T_2$ symplectically embeds into Y_K. Now suppose that $\Sigma_2 = T_1 \# T_2$ is the genus two symplectic submanifold of self-intersection zero sitting inside of Y_K. Let $(m, x, \gamma_1, \gamma_2)$ be the generators of $H_1(\Sigma_2)$ under the inclusion-induced homomorphism (here we use the same letters γ_1 and γ_2 to denote the generators of the fiber coming from the second copy of $M_K \times S^1$). We choose the involution diffeomorphism $\phi : T_1 \# T_2 \longrightarrow T_1 \# T_2$ of the Σ_2 which induces the map on first homology according to the following rule: $\phi_*(m') = \gamma_1$, $\phi_*(\gamma_1{}') = m$, $\phi_*(x') = \gamma_2$ and $\phi_*(\gamma_2{}') = x$. Next, we take the fiber sum of two copies of Y_K along the genus two surface Σ_2 and denote the new symplectic manifold as X_K, i.e. $X_K = Y_K \#_\phi Y_K$. We will show that the new manifold X_K has trivial first homology and has the same integral cohomology as $S^2 \times S^2$. Consequently, $e(X_K) = 4$, $\sigma(X_K) = 0$, $c_1{}^2(X_K) = 8$, and $\chi_h(X_K) = 1$. We will compute $H_1(X_K, \mathbb{Z})$

(also $H_2(X_K, \mathbb{Z})$) by using Mayer-Vietoris sequence and then by directly computing the fundamental group of X_K.

Lemma 4.1. *$H_1(X_K, \mathbb{Z}) = 0$ and $H_2(X_K, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$.*

Proof. We use Mayer-Vietoris sequence to compute the homology of $X_K = Y_K \#_\phi\ Y_K$. Let $Y_1 = Y_2 = Y_K \setminus \nu\Sigma_2$. By applying the reduced Mayer-Vietoris sequence to the triple (X_K, Y_1, Y_2), we have the following long exact sequence

$$\begin{aligned} \cdots \quad \longrightarrow \quad & H_2(S^1 \times \Sigma_2, \mathbb{Z}) \longrightarrow H_2(Y_1, \mathbb{Z}) \oplus H_2(Y_2, \mathbb{Z}) \longrightarrow H_2(X_K, \mathbb{Z}) \\ \longrightarrow \quad & H_1(S^1 \times \Sigma_2, \mathbb{Z}) \longrightarrow H_1(Y_1, \mathbb{Z}) \oplus H_1(Y_2, \mathbb{Z}) \longrightarrow H_1(X_K, \mathbb{Z}) \longrightarrow 0 \end{aligned}$$

The simple computation by Kunneth formula yields $H_1(\Sigma_2 \times S^1, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = < \lambda > \oplus < m > \oplus < x > \oplus < \gamma_1 > \oplus < \gamma_2 >$. Also, we have $H_1(Y_1, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} = < m > \oplus < x >$ and $H_1(Y_2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} = < m' > \oplus < x' >$.

Let ϕ_* and δ denote the last two arrows in the long exact sequence above. Because the way the gluing map ϕ is defined, the essential homology generators will map to the trivial ones. Thus, we have $\phi_*(m) = \phi_*(x) = \phi_*(m') = \phi_*(x') = 0$. Since $Im(\phi_*) = Ker(\delta)$, we conclude that $H_1(X_K) = Ker(\delta) = Im(\phi_*) = 0$

Next, by using the facts that $b_1 = b_3 = 0$, $b_0 = b_4 = 1$, and the symplectic sum formula for Euler characteristics, we compute $b_2 = e(Y_K) + e(Y_K) + 2 = 0 + 0 + 2 = 2$.

We conclude that $H_2(X_K, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. A basis for the second homology consists of classes of $\Sigma_2 = S$ and the new genus two surface T resulting from the second fiber sum operation (i.e. two punctured genus one surfaces glues to form a genus two surface), where $S^2 = T^2 = 0$ and $S \cdot T = 1$. Thus, the manifolds obtained by the above construction have intersection form H, so they are spin 4-manifolds. $\square$

Lemma 4.2. *$e(X_K) = 4$, $\sigma(X_K) = 0$, ${c_1}^2(X_K) = 8$, and $\chi_h(X_K) = 1$.*

Proof. Using the lemma 2.8, we have $e(X_K) = 2e(Y_K) + 4$, $\sigma(X_K) = 2\sigma(Y_K)$, ${c_1}^2(X_K) = 2{c_1}^2(Y_K) + 8$, and $\chi_h(X_K) = 2\chi_h(Y_K) + 1$. Since $e(Y_K) = 0$, $\sigma(Y_K) = 0$, ${c_1}^2(Y_K) = 0$ and $\chi_h(Y_K) = 0$, the result follows. $\square$

Since our basic building block $M_K \times S^1$ is a minimal symplectic 4-manifold, it follows from Usher's Theorem that the symplectic manifolds Y_K and X_K are both minimal.

4.1. Fundamental Group Computation for Trefoil

4.1.1. Step 1: Fundamental Group of $M_K \times S^1$

In this section we will assume that K is the trefoil. The case when K is the figure eight can be treated similarly.

Let a and b denote the Wirtinger generators of the trefoil knot. Then the group of K has the following presentation

$$\pi_1(S^3 \setminus \nu K) \ = \langle a, b \mid aba = bab\rangle \ = \langle u, v \mid u^2 = v^3\rangle$$

where $u = bab$ and $v = ab$. By Lemma 2.2, the homotopy classes of the meridian and the longitude of the trefoil are given as follows: $m = uv^{-1} = b$ and $l = u^2(uv^{-1})^{-6} = ab^2ab^{-4}$. Notice that $\gamma_1 = a^{-1}b$ and $\gamma_2 = b^{-1}aba^{-1}$ generate the image of the fundamental group of the Seifert surface of K under the inclusion-induced homomorphism. Let M_K denote the result of 0-surgery on K.

Lemma 4.3.

$$\begin{aligned}\pi_1(M_K \times S^1) &= \pi_1(M_K) \oplus \mathbb{Z}\\ &= \langle a, b, x \mid aba = bab,\ ab^2ab^{-4} = 1,\ [a, x] = [b, x] = 1\rangle.\end{aligned}$$

Proof. Notice that the fundamental group of M_K is obtained from the knot group of the trefoil by adjoining the relation $l = u^2(uv^{-1})^{-6} = ab^2ab^{-4} = 1$. Thus, we have the presentation given above.

□

4.1.2. Step 2: Fundamental Group of Y_K

Next, we take the two copies of the manifold $M_K \times S^1$. In the first copy, take a tubular neighborhood of the torus section $b \times x$, remove it from $M_K \times S^1$ and denote the resulting manifold as C_S. In the second copy, we remove a tubular neighborhood of the fiber F and denote the complement by C_F.

Lemma 4.4. *Let C_S be the complement of a neighborhood of a section in $M_K \times S^1$. Then we have*

$$\pi_1(C_S) = \langle a, b, x \mid aba = bab,\ [a, x] = [b, x] = 1\rangle.$$

Proof. Note that $C_S = (M_K \setminus \nu(b)) \times S^1 = (S^3 \setminus \nu K) \times S^1$. □

Lemma 4.5. *Let C_F be the complement of a neighborhood of a fiber in $M_K \times S^1$. Then we have*

$$\begin{aligned}\pi_1(C_F) = \langle \gamma_1', \gamma_2', d, y \quad | \quad & [\gamma_1', \gamma_2'] = [y, \gamma_1'] = [y, \gamma_2'] = 1,\\ & d\gamma_1' d^{-1} = \gamma_1'\gamma_2',\ d\gamma_2' d^{-1} = (\gamma_1')^{-1}\rangle.\end{aligned}$$

Proof. To compute the fundamental group of C_F, we will use the following observation: C_F is homotopy equivalent to a torus bundle over a wedge of two circles. The generators d and y do not commute in the fundamental group of C_F. Also, the monodromy along the circle y is trivial whereas the monodromy along the circle d is the same as the monodromy of M_K. Notice that $\gamma_1' = c^{-1}d$, $\gamma_2' = d^{-1}cdc^{-1}$ and $cdc = dcd$. Thus, we have
$d\gamma_1'd^{-1} = dc^{-1}dd^{-1} = dc^{-1} = \gamma_1'\gamma_2'$, and $d\gamma_2'd^{-1} = dd^{-1}cdc^{-1}d^{-1} = d^{-1}c = (\gamma_1')^{-1}$. □

Lemma 4.6. *Let Y_K be the symplectic sum of two copies of $M_K \times S^1$, identifying a section in one copy with a fiber in the other copy. If the gluing map ψ satisfies $\psi_*(x) = \gamma_1'$ and $\psi_*(b) = \gamma_2'$, then*

$$\begin{aligned}
\pi_1(Y_K) \;&=\; \langle a, b, x, \gamma_1', \gamma_2', d, y \mid aba = bab,\ [x,a] = [x,b] = 1,\\
&\quad [\gamma_1', \gamma_2'] = [y, \gamma_1'] = [y, \gamma_2'] = 1,\ d\gamma_1'd^{-1} = \gamma_1'\gamma_2',\ d\gamma_2'd^{-1} = (\gamma_1')^{-1},\\
&\quad x = \gamma_1',\ b = \gamma_2',\ ab^2ab^{-4} = [d,y]\rangle\\
&=\; \langle a, b, x, d, y \mid aba = bab,\ [x,a] = [x,b] = 1,\\
&\quad [y,x] = [y,b] = 1,\ dxd^{-1} = xb,\ dbd^{-1} = x^{-1},\ ab^2ab^{-4} = [d,y]\rangle.
\end{aligned}$$

Proof. By Van Kampen's Theorem, $\pi_1(Y_K) = \pi_1(C_S) * \pi_1(C_F)/\pi_1(T^3)$. One circle factor of T^3 is identified with the longitude of K on one side and the meridian of the torus fiber in $M_K \times S^1$ on the other side. This gives the last relation. □

Inside Y_K, we can find a genus 2 symplectic submanifold Σ_2 which is the internal sum of a punctured fiber in C_S and a punctured section in C_F. The inclusion-induced homomorphism maps the standard generators of $\pi_1(\Sigma_2)$ to $a^{-1}b$, $b^{-1}aba^{-1}$, d and y.

Lemma 4.7. *There are nonnegative integers m and n such that*

$$\begin{aligned}
\pi_1(Y_K \setminus \nu\Sigma_2) \;&=\; \langle a, b, x, d, y;\ g_1, \ldots, g_m \mid aba = bab,\\
&\quad [y,x] = [y,b] = 1,\ dxd^{-1} = xb,\ dbd^{-1} = x^{-1},\\
&\quad ab^2ab^{-4} = [d,y],\ r_1 = \cdots = r_n = 1,\ r_{n+1} = 1\rangle,
\end{aligned} \tag{1}$$

where the generators $g_1, \ldots, g_m$ and relators $r_1, \ldots, r_n$ all lie in the normal subgroup N generated by the element $[x,b]$, and the relator r_{n+1} is a word in x, a and elements of N. Moreover, if we add an extra relation $[x,b] = 1$ to (1), then the relation $r_{n+1} = 1$ simplifies to $[x,a] = 1$.

Proof. This follows from Van Kampen's Theorem. Note that $[x,b]$ is a meridian of Σ_2 in Y_K. Hence setting $[x,b] = 1$ should turn $\pi_1(Y_K \setminus \nu\Sigma_2)$ into $\pi_1(Y_K)$. Also note that $[x,a]$ is the boundary of a punctured section in $C_S \setminus \nu(\text{fiber})$, and is no longer trivial in $\pi_1(Y_K \setminus \nu\Sigma_2)$. By setting $[x,b] = 1$, the relation $r_{n+1} = 1$ is to turn into $[x,a] = 1$.

It remains to check that the relations in $\pi_1(Y_K)$ other than $[x,a] = [x,b] = 1$ remain the same in $\pi_1(Y_K \setminus \nu\Sigma_2)$. By choosing a suitable point $\theta \in S^1$ away from the image of the fiber that forms part of Σ_2, we obtain an embedding of the knot complement

$(S^3 \setminus \nu K) \times \{\theta\} \hookrightarrow C_S \setminus \nu(\text{fiber})$. This means that $aba = bab$ holds in $\pi_1(Y_K \setminus \nu\Sigma_2)$. Since $[\Sigma_2]^2 = 0$, there exists a parallel copy of Σ_2 outside $\nu\Sigma_2$, wherein the identity $ab^2ab^{-4} = [d, y]$ still holds. The other remaining relations in $\pi_1(Y_K)$ are coming from the monodromy of the torus bundle over a torus. Since these relations will now describe the monodromy of the punctured torus bundle over a punctured torus, they hold true in $\pi_1(Y_K \setminus \nu\Sigma_2)$. □

4.1.3. Step 3: Fundamental Group of X_K

Now take two copies of Y_K. Suppose that the fundamental group of the second copy has e, f, z, s, t as generators, and the inclusion-induced homomorphism in the second copy maps the generators of $\pi_1(\Sigma_2)$ to $e^{-1}f$, $f^{-1}efe^{-1}$, s and t. Let X_K denote the symplectic sum of two copies of Y_K along Σ_2, where the gluing map ψ maps the generators as follows:

$$\psi_*(a^{-1}b) = s,\ \psi_*(b^{-1}aba^{-1}) = t,\ \psi_*(d) = e^{-1}f,\ \psi_*(y) = f^{-1}efe^{-1}.$$

Lemma 4.8. *There are nonnegative integers m and n such that*

$$\begin{aligned}
\pi_1(X_K) \quad = \quad & \langle a, b, x, d, y;\ e, f, z, s, t;\ g_1, \ldots, g_m;\ h_1, \ldots, h_m \mid \\
& aba = bab,\ [y, x] = [y, b] = 1, \\
& dxd^{-1} = xb,\ dbd^{-1} = x^{-1},\ ab^2ab^{-4} = [d, y], \\
& r_1 = \cdots = r_n = r_{n+1} = 1,\ r'_1 = \cdots = r'_n = r'_{n+1} = 1, \\
& efe = fef,\ [t, z] = [t, f] = 1, \\
& szs^{-1} = zf,\ sfs^{-1} = z^{-1},\ ef^2ef^{-4} = [s, t], \\
& d = e^{-1}f,\ y = f^{-1}efe^{-1},\ a^{-1}b = s,\ b^{-1}aba^{-1} = t, [x, b] = [z, f]\rangle,
\end{aligned}$$

where g_i, h_i $(i = 1, \ldots, m)$ and r_j, r'_j $(j = 1, \ldots, n)$ all lie in the normal subgroup N generated by $[x, b] = [z, f]$. Moreover, r_{n+1} is a word in x, a and elements of N, and r'_{n+1} is a word in z, e and elements of N.

Proof. This is just a straightforward application of Van Kampen's Theorem and Lemma 4.7. □

Remark 1. Notice that the abelianization of $\pi_1(X_K)$ is trivial. Also, different symplectic cohomology $S^2 \times S^2$'s can be obtained if we use other genus 1 fibered knot, the figure eight knot, or a combination of both (the trefoil or the figure eight knot) in our construction. Using computational software [GAP], we show that these manifolds have different fundamental groups. In addition, considering an infinite family of non-fibered genus one twist knots, we also obtain an infinite family of cohomology $S^2 \times S^2$. This family of cohomology $S^2 \times S^2$'s will not be symplectic anymore and can be distinguished by their Seiberg-Witten invariants. Also, using [GAP], it is not hard to verify that their fundamental groups are different as well.

Question 1. Are there two distinct genus 1 knots K and $K^{'}$ such that $\pi_1(X_K)$ is isomorphic to $\pi_1(X_{K'})$?

4.2. Seiberg-Witten invariants for manifold X_K

Let C be a basic class of the manifold X_K. We can write C as a linear combination of S and T, i.e. $C = aS + bT$. X_K is a symplectic manifold and has simple type. So for any basic class C, $C^2 = 3\sigma(X_K) + 2e(X_K) = 8$. It follows that $2ab = 8$. Next we apply the adjunction inequality to S and T to get $2g(S) - 2 \geq [S]^2 + |C(S)|$ and $2g(T) - 2 \geq [T]^2 + |C(T)|$. These gives us two more restriction on a and b : $2 \geq |b|$ and $2 \geq |a|$. Thus, it follows that $C = \pm(2S + 2T)$, which is $\pm$ the canonical class $K_{X_K} = 2S + 2T$ of X_K. Notice that, since $b_2^-(X_K) \leq 9$ and $(-K_{X_K}) \cdot \omega < 0$, we have a well defined $\mathrm{SW}^o_{X_K}$. Now it follows from the theorems of Section 3 that $\mathrm{SW}^0_{X_K}(-K_{X_K}) = \mathrm{SW}^-_{X_K}(-K_{X_K}) = \pm 1$.

5. Symplectic manifolds cohomology equivalent to $\#_{(2g-1)}(S^2 \times S^2)$

In this section, we will modify the construction above to get an infinite family of symplectic 4-manifolds cohomology equivalent to $\#_{(2g-1)}(S^2 \times S^2)$ for any $g \geq 2$.

Let K' denote a genus g fibered knot in S^3 and m a meridianal circle to K'. We first perform 0-framed surgery on K' and denote the resulting 3-manifold by $Z_{K'}$. The 4-manifold $Z_{K'} \times S^1$ is a Σ_g bundle over the torus and has the same integral homology as $T^2 \times S^2$. Since K' is a fibered knot, $Z_{K'} \times S^1$ is a symplectic manifold. Again, there is a torus section $m \times S^1 = T_m$ of this fibration. The first homology of $Z_{K'} \times S^1$ is generated by the standard first homology generators of this torus section. The standard homology generators of the fiber F, which we denote as $\alpha_2, \beta_2, \cdots, \alpha_{g+1}$, and β_{g+1} of the given bundle is trivial in the homology. The section T_m has zero self-intersection and the its neighborhood in $Z_{K'} \times S^1$ has a canonical identification with $T_m \times D^2$.

We form a twisted fiber sum of the manifold $M_K \times S^1$ with $Z_{K'} \times S^1$ where we identify the fiber of the first fibration to the section of other. Let $W_{KK'}$ denote the new manifold $W_{KK'} = M_K \times S^1 \#_{F=T_{m'}}\ Z_{K'} \times S^1$. Let us remark that the manifold $W_{KK'}$ is obtained from $M_K \times S^1$ by knot surgery of Fintushel-Stern [FS1] along knot $K^{'}$. Again, it follows from Gompf's theorem [Go] that $W_{KK'}$ is symplectic.

Let T_1 be the section of the $M_K \times S^1$ which we discussed earlier and Σ_g be the fiber of the $Z_{K'} \times S^1$. Then $T_1 \# \Sigma_g$ embeds in $W_{KK'}$ and has self-intersection zero. Now suppose that $W_{KK'}$ is the symplectic 4-manifold given, and $\Sigma_{g+1} = T_1 \# \Sigma_g$ is the genus $g+1$ symplectic submanifold of self-intersection 0 sitting inside of $W_{KK'}$. Let α_1, β_1, α_2, β_2, $\ldots$, α_{g+1}, and β_{g+1} be the generators of the first homology of the surface $T_1 \# \Sigma_g = \Sigma_{g+1}$. Let $\psi : T^2 \# \Sigma_g \to \Sigma_g \# T^2$ be any diffeomorphism of the genus $g+1$ surface that changes the generators of the first homology according to the following rule: $\psi_*(\alpha_1) = \alpha'_{g+1}$, $\psi_*(\beta_1) = \beta'_{g+1}$, $\psi_*(\alpha_{g+1}) = \alpha'_1$, and $\psi_*(\beta_{g+1}) = \beta'_1$. Take the fiber sum along this genus $g+1$ surface Σ_{g+1} and denote the resulting symplectic manifold as $V_{KK'}$. The new manifold $V_{KK'} = W_{KK'} \#_{\psi}\ W_{KK'}$ has trivial first homology and has the same homology of $\#_{(2g-1)} S^2 \times S^2$.

Lemma 5.1. *$H_1(V_{KK'}, \mathbb{Z}) = 0$ and $H_2(V_{KK'}, \mathbb{Z}) = \oplus_{2(2g-1)} \mathbb{Z}$.*

Proof. The proof is similar to genus one case and can be obtained by applying Mayer-Vietoris sequence.

□

Note that $H_2(V_{KK'}, \mathbb{Z})$ has rank $4g-2$. A basis for the second homology consist of the $[S]$ and $[\Sigma]$, where S is the genus $g+1$ surface Σ_{g+1} and Σ is the genus two surface Σ_2 resulting from the last fiber sum operation, $(2g-2)$ Lagrangian tori R_i and $(2g-2)$ associated dual Lagrangian tori V_i. In the manifold $V_{KK'}$ these classes contribute $2g-2$ new hyperbolic pairs. Thus, the manifolds obtained by the above construction have intersection form $\oplus_{2g-1} H$. Notice that $b_2^+(V_{KK'}) = b_2^-(V_{KK'}) = 2g-1$

Lemma 5.2. *$e(V_{KK'}) = 4g$, $\sigma(V_{KK'}) = 0$, ${c_1}^2(V_{KK'}) = 8g$, and $\chi_h(V_{KK'}) = g$.*

Proof. Using the lemma 2.8, we have $e(V_{KK'}) = 2e(W_{KK'}) + 4g$, $\sigma(V_{KK'}) = 2\sigma(W_{KK'})$, ${c_1}^2(V_{KK'}) = 2{c_1}^2(W_{KK'}) + 8g$ and $\chi_h(V_{KK'}) = 2\chi_h(W_{KK'}) + g$. Since $e(W_{KK'}) = 0$ and $\sigma(W_{KK'}) = 0$, our result follows. □

Remark 2. Using Lemma 2.2 and applying steps of Section 4.1, one can compute $\pi_1(V_{KK'})$ when K is the trefoil and K' is the (p,q) torus knot.

Question 2. Are there two distinct genus $g \geq 2$ knots K' and K'' such that $\pi_1(V_{KK'})$ is isomorphic to $\pi_1(V_{KK''})$?

5.1. Seiberg-Witten invariants for manifold $V_{KK'}$

Let C be a basic class of the manifold $V_{KK'}$. Let us write C as a linear combination of S, Σ, tori R_i, $i = 1, ..., 2g-2$ and the dual tori V_i, $i = 1, \cdots, 2g-2$, $C = aS + b\Sigma + \sum_{i=1}^{2g-2} u_i R_i + v_i V_i$. By adjunction inequality, the intersection number of any basic class with tori R_i is zero, i.e. $C \cdot R_i = 0$, $i = 1, \cdots, 2g-2$. It follows that $Q^T v = 0$ where Q is the intersection matrix and $v = (v_1, \cdots, v_{2g-2})$. Using the fact that Q is invertible, we obtain $v_1 = \cdots = v_{2g-2} = 0$. Next, by applying the adjunction inequality again, we have $V_i \cdot C = 0$ for $i = 1, \cdots, 2g-2$. This gives rise to the system $Qu = 0$, which implies $u_1 = \cdots = u_{2g-2} = 0$. This shows that any basic class has form $C = aS + b\Sigma$. Since $V_{KK'}$ is a symplectic manifold and has simple type. So for any basic class C, $C^2 = 3\sigma(V_{KK'}) + 2e(V_{KK'}) = 8g$. It follows that $2ab = 8g$. Next we apply the adjunction inequality to S and Σ to get $2g(S) - 2 \geq [S]^2 + |C(S)|$ and $2g(\Sigma) - 2 \geq [\Sigma]^2 + |C(\Sigma)|$. These gives two more restrictions on a and b : $2g \geq |b|$ and $2 \geq |a|$. It follows that $C = \pm(2S + 2g\Sigma)$, which are $\pm$ the canonical class of the given manifold. By applying the Theorem 3.5, we conclude that the value of Seiberg-Witten invariants of these classes is ± 1.

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References

[BZ] G. Burde, H. Zieschang, *Knots*, New York: de Gruyter, 1985.

[FS1] R. Fintushel, R. Stern, *Knots, links, and 4-manifolds*, Invent. Math. **134** (1998), 363-400.

[FS2] R. Fintushel, R. Stern, *Families of simply connected 4-manifolds with the same Seiberg-Witten invariants*, Topology, **8**(2004), 949–968.

[GAP] *GAP - Groups, Algorithms, and Programming*, Version 4.3; 2002 **(http://www.gap-system.org)**

[Go] R. E. Gompf: A new construction of symplectic manifolds, *Ann. of Math.* **142** (1995), 527–595.

[KM] P. Kronheimer and T. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Letters **1**, (1994) 797-808.

[LL] T. J. Li and A. Liu, *General wall crossing formula*, Math. Res. Letters **2**, (1995) 797-810.

[Mu] D. Mumford, *An Algebraic surfaces with K ample and $K^2 = 9$, $p_g = q = 0$*, American Journal of Mathematics, **101** (1979), 233-244.

[OS] P. Ozsváth, Z. Szabó *The Symplectic Thom conjecture*, Ann. Math. **151** (2000), 93-124.

[P] R. Pardini, *The Classification of double planes of general type with $K^2 = 8$, $p_g = q = 0$*, Journal of Algebra, **259** (2003), 95-118.

[R] D. Rolfsen, *Knots and Links*, Publish or Perish 1990.

[S] J. Stallings, *On fibering certain 3-manifolds*, Proc. Univ. Georgia Inst., Prentice-Hall (1961) 95-100.

[Sz] Z. Szabó, *Exotic 4-manifolds with $b_2^+ = 1$*, Math. Res. Letters **3** (1996), 731–741.

[T] C. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Letters **1** (1994), 809–822.

[U] M. Usher: *Minimality and Symplectic Sums*, Internat. Math. Res. Not. 2006 preprint, http://front.math.ucdavis.edu/math.SG/0606543

[W] E. Witten, *Monopoles and four-manifolds*, Math. Res. Letters **1** (1994), 769–796.

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Some remarks on cabling, contact structures, and complex curves

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Abstract. We determine the relationship between the contact structure induced by a fibered knot, $K \subset S^3$, and the contact structures induced by its various cables. Understanding this relationship allows us to classify fibered cable knots which bound a properly embedded complex curve in the four-ball satisfying a genus constraint. This generalizes the well-known classification of links of plane curve singularities.

1. Introduction

A well-known construction of Thurston and Winkelnkemper [ThuWin] associates a contact structure to an open book decomposition of a three-manifold. This allows us to talk about *the contact structure associated to a fibered knot*. Here, a fibered knot is a pair, $(F, K) \subset Y$, such that $Y - K$ admits the structure of a fiber bundle over the circle with fibers isotopic to F and $\partial F = K$. We denote the contact structure associate to a fibered knot by $\xi_{F,K}$ or, when the fiber surface is unambiguous, by ξ_K.

Thus any operation on knots (or Seifert surfaces) which preserves the property of fiberedness induces an operation on contact structures. For instance, one can consider the Murasugi sum operation on surfaces-with-boundary (see [Gab] for definition). In this case, a result of Stallings [Sta] indicates that the Murasugi sum of two fiber surfaces is also fibered (a converse to this was proved by Gabai [Gab]). The effect on contact structures is given by a result of Torisu:

Theorem 1.1. (*Theorem* 1.3 *of* [Tor].) *Let* $(F_1, \partial F_1) \subset Y_1$ $(F_2, \partial F_2) \subset Y_2$ *be two fiber surfaces, and let* $(F_1 * F_2, \partial(F_1 * F_2)) \subset Y_1 \# Y_2$ *denote any Murasugi sum. Then*

$$\xi_{F_1 * F_2, \partial(F_1 * F_2)} \simeq \xi_{F_1, \partial F_1} \# \xi_{F_2, \partial F_2},$$

where the right-hand side denotes the connected sum of contact structures.

Another operation which preserves fiberedness is cabling, whose definition we now recall. For a knot $K \subset Y$, there is an identification of a tubular neighborhood of K with a solid torus, $S^1 \times D^2$, and a *cable knot* is the image of a torus knot living on $\partial(S^1 \times D^2)$ under this identification. Note that the identification depends on a choice of longitude for K, and we choose the canonical longitude coming from a Seifert surface for K. We denote

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the (p, q) cable of K by $K_{p,q}$, with q indicating the linking number of $K_{p,q}$ with K i.e. the number of times the torus knot wraps around the meridian of the tubular neighborhood of K. Thus $U_{p,q} = T_{p,q}$ where U is the unknot and $T_{p,q}$ is the (p, q) torus knot. Throughout we will consider knots, and hence p and q will be relatively prime. We also restrict to the case $p > 0$. There is no loss of information in doing this, since $K_{p,q} = K_{-p,-q}$ as unoriented knots, and none of our arguments are sensitive to orientation.

In the case now that K is fibered, $K_{p,q}$ will also be fibered. This follows, for instance, from [Sta] though the fibration of $Y - K_{p,q}$ can easily be visualized. Thus it makes sense to ask how the contact structure associated to a fibered knot changes, if at all, upon cabling. The purpose of this note is to answer this question for the case of knots in S^3. To state the result, recall that contact structures on S^3 are in bijection with the integers plus a distinguished element [Eli1, Eli2]:

$$\{\text{Isotopy classes of contact structures on } S^3\} \leftrightarrow \mathbb{Z} \cup \{\bullet\}.$$

The contact structure corresponding to the distinguished element, which we denote ξ_{std}, is the unique tight contact structure. It is obtained as the field of complex lines tangent to the three-sphere, viewed as the boundary of the Stein four-ball in $\mathbb{C}^2$. The other, overtwisted, contact structures correspond to homotopy classes of two-plane fields on S^3 [Eli1], and are determined uniquely by their Hopf invariant, $h(\xi) \in \mathbb{Z}$. We let ξ_i denote the overtwisted contact structure with $h(\xi_i) = i$. In these terms, we have:

Theorem 1.2. *Let $K \subset S^3$ be a fibered knot of Seifert genus g, and let $K_{p,q}$ denote its (p, q) cable. Then for any $p > 0$, we have*

1. *If $q > 0$, then $\xi_{K_{p,q}} \simeq \xi_K$.*
2. *If $q < 0$, then $\xi_{K_{p,q}} \simeq \xi_K \# \xi_{(1-p)(2g-q-1)}$*

In particular, $\xi_{K_{p,q}} \simeq \xi_{std}$ if and only if $\xi_K \simeq \xi_{std}$ and $q > 0$.

Cable knots play a prominent role in the classification of plane curve singularities, where certain iterated torus knots naturally arise (an iterated torus knot is an iterated cable of the unknot). To explain this, let $f(z, w) \in \mathbb{C}[z, w]$ be a complex polynomial with isolated singularity at the origin $(0, 0) \in \mathbb{C}^2$. Recall that the *link of the singularity of* $f(z, w)$ is defined to be the knot or link, K_f, which comes from the intersection

$$K_f = \{(z, w) \in \mathbb{C}^2 | f(z, w) = 0\} \cap S^3_\epsilon,$$

where $S^3_\epsilon = \{(z, w) \in \mathbb{C}^2 | |z|^2 + |w|^2 = \epsilon\}$ is the unit sphere of radius ϵ in $\mathbb{C}^2$. It is straightforward to show that the isotopy type of K_f as a knot or link is well-defined, provided ϵ is sufficiently small. The classification of links of plane curve singularities shows that the only knots which arise in this way are iterated cables of the unknot whose cabling parameters satisfy a positivity condition (see [EisNeu] for a discussion). For example, the (p', q') cable of the (p, q) torus knot is the link of a singularity precisely when $q' \geq p'pq + 1$.

An important property of links of singularities is that they bound smooth complex curves of genus equal to their Seifert genus. Indeed, a small perturbation of f produces a smooth complex curve which can be deformed into the three-sphere without singularities. This surface is a Seifert surface for K_f which can be seen to be a fiber in a fibration of $S^3 - K_f$ (see [Mil] for more details).

While the iterated torus knots which arise as links of singularities are classified, one can relax the condition that the radius of the three-sphere S^3_ϵ be sufficiently small. In this case, the four-ball may contain multiple singular points of a given complex curve. The general case is handled by fixing the radius of the three-sphere and asking:

Question 1.3. Which iterated torus knots arise as the intersection of a complex curve $V_f \subset \mathbb{C}^2$ with S^3_1, the unit radius three-sphere? Of these, which among them have a Seifert surface which can be isotoped into the four-ball to the piece of the curve, $V_f \cap B^4$?

Drawing on our previous work [Hed1], Theorem 1.2 answers the second part of the question as a corollary.

Corollary 1.4. *Let $p > 0$, and let K be a fibered knot. Then $K_{p,q}$ has a Seifert surface which is isotopic to a piece of a complex curve $V_f \cap B^4$ if and only if*

- *K has a Seifert surface which is isotopic to a piece of a complex curve* and
- *$q > 0$*

In particular, the fiber surface of an iterated torus knot is isotopic to a piece of a complex curve if and only if all the cabling coefficients are positive.

Remark 1.5. The corollary answers a question of Baader and Ishikawa (Question 6.10 of [BaaIsh]). They asked whether there exists a quasipositive fiber surface, other than a disk, from which we cannot deplumb a Hopf band (see Figure 1 and the associated discussion in Section 2 for relevant definitions). The answer is yes, and an example is provided by the $(2, 1)$ cable of the right-handed trefoil. The proof of the corollary shows that it bounds a quasipositive fiber surface, but it does not deplumb a Hopf band by work of Melvin and Morton [MelMor].

We prove Theorem 1.2 and Corollary 1.4 in the next section. Theorem 1.2 will be proved by examining the knot Floer homology invariants of cable knots, which we studied in [Hed2, Hed3]. For fibered knots in the three-sphere, these invariants completely determine the contact structure induced by the knot. Corollary 1.4 will follow from a result in [Hed1] which connects a particular Floer invariant or, equivalently, the tightness of the contact structure induced by K, with complex curves.

While Theorem 1.2 cannot be extended in general using the techniques of the present paper, it can be done in the case of three-manifolds, Y^3, for which the Ozsváth-Szabó contact invariant classifies contact structures on Y^3 (e.g. lens spaces). However, it seems reasonable to expect that the theorem holds for arbitrary three-manifolds. That is, positive cabling should not change the contact structure associated to a fibered knot, while negative cabling has the effect of taking the connected sum with $(S^3, \xi_{(1-p)(2g-q-1)})$. Indeed, this could probably be proved with the rudiments of convex surface theory. We content ourselves here with the case of S^3 since the results follow easily from our previous work on cables. Moreover, our proof serves to highlight the rich geometric content of the knot Floer homology invariants. In particular, we found it surprising that knot Floer homology could be useful in a result such as Corollary 1.4.

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2. Proof of Theorem 1.2

In this section we prove Theorem 1.2. This is accomplished in two steps. The first is to determine the effect of cabling on the Hopf invariant. The next is to determine whether a cable knot induces the tight contact structure, ξ_{std}. Since contact structures on S^3 are determined by their Hopf invariant and whether they are tight, this will complete the proof. Both tasks can be accomplished with an understanding of the knot Floer homology invariants of cable knots, which we established in [Hed2, Hed3]. The knot Floer homology invariants of a fibered knot $K \subset S^3$, in turn, determine the Hopf invariant of ξ_K and whether ξ_K is tight.

2.1. Contact structures and knot Floer homology of fibered knots

We begin by briefly describing the algebraic structure of the knot Floer homology invariants and how these invariants determine the contact structure associated to a fibered knot $K \subset S^3$.

To a closed oriented three-manifold, Y, and Spin^c structure, $\mathfrak{s}$, Ozsváth and Szabó introduced a chain complex $\widehat{CF}(Y, \mathfrak{s})$ [OzsSza1]. A null-homologous knot (Y, K) induces a filtration $\mathcal{F}(Y, K, i)$ of this chain complex, i.e. there is an increasing sequence of subcomplexes:

$$0 = \mathcal{F}(Y, K, -i) \subseteq \mathcal{F}(Y, K, -i+1) \subseteq \ldots \subseteq \mathcal{F}(Y, K, n) = \widehat{CF}(Y, \mathfrak{s}).$$

For the definition of this filtration see [OzsSza2, Ras]. The homology of the successive quotients in this filtration are the *knot Floer homology groups of* (Y, K). In the case of the three-sphere, we denote these groups by

$$\widehat{HFK}_*(K, i) \cong H_*\left(\frac{\mathcal{F}(S^3, K, i)}{\mathcal{F}(S^3, K, i-1)}\right)$$

The knot Floer homology groups have the following symmetries (Propositions 3.7 and 3.10 of [OzsSza2], respectively):

$$\widehat{HFK}_*(K, i) \cong \widehat{HFK}_{-*}(\overline{K}, -i). \qquad (1)$$

$$\widehat{HFK}_*(K, i) \cong \widehat{HFK}_{*-2i}(K, -i). \qquad (2)$$

In the first equation, $\overline{K}$ denotes the reflection of K (i.e. in a projection, change all crossings simultaneously or, equivalently, consider the image of K under an orientation-reversing involution of S^3). In [OzsSza3], Ozsváth and Szabó proved the following:

Proposition 2.1. *Let $K \subset S^3$ be a fibered knot, and let $g(K) = g$ denote its Seifert genus. Then*

$$\widehat{HFK}_*(K, g) \cong \begin{cases} \mathbb{Z} & * = -h(\xi_K) \\ 0 & \textit{otherwise.} \end{cases}$$

where $h(\xi_K)$ is the Hopf invariant of the contact structure induced by K.

Remark 2.2. This proposition is a combination of Theorem 1.1 and Proposition 4.6 of [OzsSza3]. More precisely, Theorem 1.1 shows that

$$\widehat{HFK}_*(-S^3, K, -g) \cong H_*(\mathcal{F}(-S^3, K, -g)) \cong \mathbb{Z},$$

and Proposition 4.6 shows that the grading of this group is $h(\xi_K)$. Equation (1) then determines the effect of reflection on knot Floer homology (i.e. $(-S^3, K) = (S^3, \overline{K})$). Note that each of the aforementioned results generalize to knots in manifolds other than S^3.

This yields the following property of $h(\xi_K)$, which can also be found in [Rud1].

Corollary 2.3. *The Hopf invariant satisfies $h(\xi_K) = -h(\xi_{\overline{K}}) - 2g$, where g is the genus of K.*

Proof. Combine Proposition 2.1 with Equations (1) and (2). □

The above proposition indicates that the grading on knot Floer homology captures the homotopy class of the contact structure. In the case of the three-sphere, it remains to understand whether a contact structure is tight or overtwisted. To this end, recall that the Floer homology of the three-sphere satisfies

$$\widehat{HF}_*(S^3) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we can define the following numerical invariant of K,[OzsSza4, Ras]:

$$\tau(K) = \min\{j \in \mathbb{Z} | i_* : H_*(\mathcal{F}(S^3, K, j)) \longrightarrow \widehat{HF}(S^3) \cong \mathbb{Z} \text{ is nontrivial}\}.$$

For fibered knots, $\tau(K)$ determines whether ξ_K is tight or overtwisted. Moreover, it determines whether K arises as the boundary of a properly embedded complex curve $V_f \subset B^4$ satisfying $g(V_f) = g(K)$.

Proposition 2.4. (*Proposition* 2.1 *of* [Hed1].) *Let $K \subset S^3$ be a fibered knot and F its fiber surface. Then the following are equivalent:*

1. *K satisfies $g(K) = \tau(K)$.*
2. *$\xi_K \simeq \xi_{std}$.*
3. *K is strongly quasipositive (with F a quasipositive Seifert surface).*
4. *F is isotopic to a properly embedded complex curve $(V_f, K) \subset (B^4, S^3)$.*

Remark 2.5. A strongly quasipositive knot is a knot which bounds a so-called quasipositive Seifert surface. Quasipositive Seifert surfaces, in turn, are those surfaces-with-boundary which are obtained from parallel disks by attaching positive bands. See Figure 1 for examples, and [Hed1] for further details. We have added (4) above, to the equivalences of [Hed1]. However, (3) implies (4) by work of Rudolph [Rud3], and (4) implies (1) by Theorem 1.5 of [Hed1].

Thus, to determine the effect of cabling on contact structures, it will suffice to understand the behavior of the grading of the "top group" of knot Floer homology under cabling, together with the behavior of $\tau(K)$. We begin with the former.

2.2. Behavior of the Hopf invariant under cabling

In this section, we determine the effect of cabling on the Hopf invariant of a fibered knot. We prove the following:

Theorem 2.6. *Let $p > 0$, and let K be a fibered knot of genus g. Then we have*

$$h(\xi_{K_{p,q}}) = \begin{cases} h(\xi_K) & \text{for } q > 0 \\ h(\xi_K) + (1-p)(2g-q-1) & \text{for } q < 0. \end{cases}$$

Remark 2.7. This result can be derived from a theorem of Neumann and Rudolph [NeuRud1], proved in a different context. They proved a similar formula for the *enhanced Milnor number* of a fibered link, which they denote $\lambda(K)$. Our formula follows from theirs since, after tracing through the definitions of the Hopf invariant and enhanced Milnor number, one finds that $\lambda(K) = -h(\xi_K)$. Our proof, using Floer homology, is quite different in spirit and follows easily from our previous understanding of the Floer homology of cable knots.

The main tool will be the following result from [Hed2]

Proposition 2.8. (*Corollary* 1.3 *of* [Hed2].) *Let $p > 0$ and let $K \subset S^3$ be a knot of Seifert genus g. Then there exists an $N > 0$ such that for all $n > N$, the following holds:*

$$\widehat{HFK}_*\left(K_{p,pn+1},\ pg + \frac{(p-1)(pn)}{2}\right) \cong \widehat{HFK}_*(K, g)$$

Remark 2.9. Note that $pg + \frac{(p-1)(pn)}{2}$ is the genus of $K_{p,pn+1}$.

Propositions 2.1 and 2.8 determine the effect of cabling on the Hopf invariant when $q = pn + 1 \gg 0$. An observation of Neumann and Rudolph [NeuRud2] which interprets an arbitrary cable as a Murasugi sum of the $(p, \pm 1)$ cable and an appropriate torus knot will allow us to take care of the case when $q > 0$.

Proposition 2.10. (*Figure* 4.2 *of* [NeuRud2] *or Figure* 1 *of* [Rud2].) *Assume $p > 0$. Then $K_{p,q}$ is a Murasugi sum of $K_{p,\mathrm{sgn}(q)}$ and $T_{p,q}$. Here,* sgn(q) *is the function which is* 1 *if $q > 0$ and -1 otherwise.*

In the above, the Murasugi sum can be taken to be along minimal genus Seifert surfaces which, for our purposes, will be fiber surfaces.

Proof of Theorem 2.6 when $q > 0$:
For all sufficiently large $n > 0$, we have

$$h(\xi_K) = h(\xi_{K_{p,pn+1}}) = h(\xi_{K_{p,1} * T_{p,pn+1}}) = h(\xi_{K_{p,1}}) + h(\xi_{T_{p,pn+1}}) =$$

$$= h(\xi_{K_{p,1}}) + h(\xi_{T_{p,q}}) = h(\xi_{K_{p,1} * T_{p,q}}) = h(\xi_{K_{p,q}}).$$

Here, the first equality follows from Proposition 2.8, the second from Proposition 2.10, and the third from the well-known additivity of the Hopf invariant under Murasugi sum (see, for instance [NeuRud2]). The fourth equality follows from the fact that any positive torus knot has Hopf invariant 0 (which can be seen, for instance by realizing them all as plumbings of positive Hopf bands). The fifth and six equalities are an application of additivity of Hopf invariant and Proposition 2.10, respectively. □

Proof of Theorem 2.6 when $q < 0$:
For all sufficiently large $n > 0$, we have

$$\begin{aligned} h(\xi_{K_{p,-pn-1}}) &= -h(\xi_{\overline{K}_{p,pn+1}}) - 2\left(pg + \tfrac{(pn)(p-1)}{2}\right) = \\ &= -h(\xi_{\overline{K}}) - 2pg + (1-p)pn = h(\xi_K) + (1-p)(2g + pn). \end{aligned} \tag{3}$$

The first equality follows from Corollary 2.3, bearing in mind the genus of $K_{p,-pn-1}$ and the fact that $\overline{K_{p,-pn-1}} = \overline{K}_{p,pn+1}$. The second equality follows from the previous case, since $pn + 1 > 0$. The third equality is an application of Corollary 2.3.

Next we have

$$h(\xi_{K_{p,-pn-1}}) = h(\xi_{K_{p,-1}}) + h(\xi_{T_{p,-pn-1}}) = h(\xi_{K_{p,-1}}) + (1-p)(pn). \tag{4}$$

Here, the first equality comes from Proposition 2.10 and additivity of the Hopf invariant under Murasugi sums. The second equality comes from Corollary 2.3, noting that $\overline{T_{p,-pn-1}} = T_{p,pn+1}$ and $2g(T_{p,-pn-1}) = (p-1)pn$.

By the same considerations, we have

$$h(\xi_{K_{p,q}}) = h(\xi_{K_{p,-1}}) + h(\xi_{T_{p,q}}) = h(\xi_{K_{p,-1}}) + (1-p)(-q-1) \tag{5}$$

Solving Equations (3), (4), and (5) for $h(\xi_{K_{p,q}})$ yields

$$h(\xi_{K_{p,q}}) = h(\xi_K) + (1-p)(2g - q - 1),$$

as claimed. □

2.3. Determination of fibered cable knots which induce ξ_{std}

In this section, we use Proposition 2.4 to determine which cable knots induce the standard tight contact structure on S^3. We prove the following:

Theorem 2.11. *Let $p > 0$, and let K be a non-trivial fibered knot. Then $\xi_{K_{p,q}} \simeq \xi_{std}$ if and only if $\xi_K \simeq \xi_{std}$ and $q > 0$.*

Again, we rely on our previous results concerning the Floer homology of cables. In this case, we call upon Theorem 1.2 of [Hed3]

Theorem 2.12. *Let $K \subset S^3$ be a knot. Then the following inequality holds for all n,*

$$p \cdot \tau(K) + \frac{(pn)(p-1)}{2} \quad \leq \tau(K_{p,pn+1}) \leq \quad p \cdot \tau(K) + \frac{(pn)(p-1)}{2} + p - 1.$$

In the special case when K satisfies $\tau(K) = g(K)$ we have the equality,

$$\tau(K_{p,pn+1}) = p \cdot \tau(K) + \frac{(pn)(p-1)}{2}.$$

Proof of Theorem 2.11:

Recall that Proposition 2.10 shows that $K_{p,q} = K_{p,\mathrm{sgn}(q)} * T_{p,q}$. Torisu's theorem then implies that $\xi_{K_{p,q}} \simeq \xi_{K_{p,\mathrm{sgn}(q)}} \# \xi_{T_{p,q}}$. In the case that $q < -1$, $\xi_{T_{p,q}}$ is overtwisted, as indicated by the fact that $h(\xi_{T_{p,q}}) = (p-1)(-q-1)$. Thus $K_{p,q}$ is overtwisted, provided $q < -1$. Likewise, if $q \geq 1$ then $\xi_{T_{p,q}} \simeq \xi_{std}$. Thus

$$\xi_{K_{p,q}} \simeq \xi_{K_{p,1}} \# \xi_{std} \simeq \xi_{K_{p,1}}$$

whenever $q \geq 1$.

In light of these remarks, it suffices to understand $\xi_{K_{p,\pm 1}}$.

Assume first that $\xi_K \simeq \xi_{std}$. Proposition 2.4 indicates that $\tau(K) = g(K)$. The second part of Theorem 2.12 tells us that (for $n = 0$)

$$\tau(K_{p,1}) = pg(K) = g(K_{p,1}).$$

Hence, $\xi_{K_{p,1}}$ is tight, by Proposition 2.4. On the other hand, Theorem 2.6 implies

$$h(\xi_{K_{p,-1}}) = h(\xi_K) + 2g(K)(1-p) = 2g(K)(1-p) \neq 0,$$

Where the second equality follows from the fact that $\xi_K \simeq \xi_{std}$, by assumption, and hence has vanishing Hopf invariant. Thus, $\xi_{K_{p,-1}}$ is overtwisted. This proves Theorem 2.11 in the case that ξ_K is tight.

If ξ_K is not tight, then $\tau(K) < g(K)$ by Proposition 2.4 (note that $\tau(K) \leq g(K)$ by the adjunction inequality for knot Floer homology, Theorem 5.1 of [OzsSza2]). Theorem 2.12 then shows that

$$\tau(K_{p,1}) \leq p\tau(K) + p - 1 < pg(K) = g(K_{p,1}).$$

Hence $\xi_{K_{p,1}}$ is not tight, by Proposition 2.4. To deal with $K_{p,-1}$, note that $K_{p,1}$ can be changed into $K_{p,-1}$ by a sequence of $(p-1)$ crossing changes, each of which change a

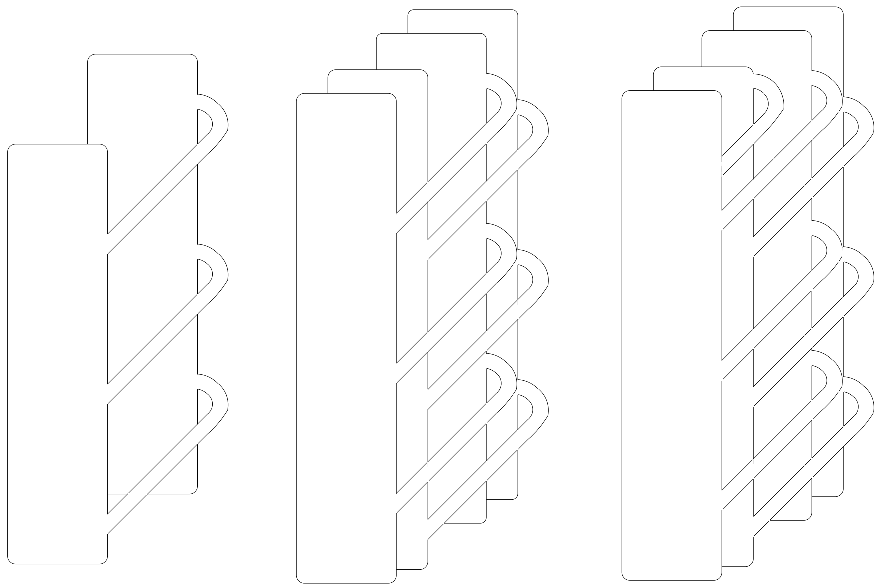

FIGURE 1. Quasipositive Seifert surfaces are surfaces obtained, like the ones pictured here, by stacking parallel disks and attaching positive bands (here, positive means that the bands turn upwards on the right). This figure depicts the manner in which we can cable a quasipositive Seifert surface. The left figure is a quasipositive Seifert surface, F, for the trefoil. The middle represents the first surface together with a parallel push-off in a product neighborhood of F. The middle surface is clearly quasipositive and its boundary is the $(2, 0)$ cable link of the trefoil. The third surface is obtained by adding a positive band to the second, and is a quasipositive Seifert surface for the $(2, 1)$ cable of the trefoil.

positive to crossing to a negative. Corollary 1.5 of [OzsSza4] indicates that $\tau(K)$ cannot increase under such crossing changes, and hence

$$\tau(K_{p,-1}) \leq \tau(K_{p,1}) < g(K_{p,-1}).$$

Thus $\xi_{K_{p,-1}}$ is not tight. This completes the proof. □

Proof of Corollary 1.4:

The corollary follows immediately from Theorem 2.11 and Proposition 2.4. Indeed, Proposition 2.4 characterizes fibered knots which induce ξ_{std} as precisely those for which the fiber surface is isotopic to a piece of complex curve. Theorem 2.11 then determines which cable knots induce ξ_{std}. □

Though we have proved the theorem and corollary, we think it is interesting to note that if K has a quasipositive Seifert surface, then a quasipositive surface for $K_{p,q}$ can be explicitly constructed for $q > 0$. Since quasipositive surfaces are isotopic to pieces of smooth complex curves in the four-ball [Rud3], this indicates that positive cabling is, in some sense, a complex operation.

To construct the aforementioned surfaces, we simply observe that we can "cable" a quasipositive surface, F, by taking p copies of each disk and band used to construct F. See Figure 1 for an illustration. More precisely, we consider a neighborhood, $F \times [0, 1]$, of $F \subset S^3$ and take the disjoint surfaces $F \times \{x_i\}$ for p distinct points $x_i \in [0, 1]$, $i = 1, .., p$. The boundary of the resulting (quasipositive) surface is the $(p, 0)$ cable link on K. To obtain the positive cable knots, we simply attach some more positive bands. Note that this method proves the "if" direction of Theorem 2.11 without appealing to the second part of Theorem 2.12. However, we could not obtain a general proof of the "only if" direction which avoided the use of Theorem 2.12.

References

[BaaIsh] S. Baader and M. Ishikawa. *Legendrian graphs and quasipositive diagrams*, *arxiv.org/abs/math/0609592*.

[EisNeu] D. Eisenbud and W. Neumann. *Three-dimensional Link Theory and Invariants of Plane Curve Singularities*, Annals of Mathematics Studies, *110*, Princeton University Press, (1985).

[Eli1] Y. Eliashberg. *Classification of overtwisted contact structures on 3-manifolds*, Inventiones Mathematicae, *98(3)*, (1989), 623–637.

[Eli2] Y. Eliashberg. *Contact 3-manifolds twenty years since J. Martinet's work*, *Ann. Inst. Fourier*, *42*, (1992), 165–192.

[Gab] D. Gabai. *The Murasugi sum is a natural geometric operation*, *Proceedings of the Special Session on Low Dimensional Topology, 87th Meeting of the AMS, (1983), 131–143.*

[Hed1] M. Hedden. *Notions of Positivity and the Ozsváth-Szabó concordance invariant*, *to appear J. Knot Theory Ramif., arXiv:math.GT/0509499.*

[Hed2] M. Hedden. *On Knot Floer homology and cabling*, Algebr. Geom. Topol., *5*, (2005), 1197–1222.

[Hed3] M. Hedden. *On knot Floer homology and cabling II*, *In preparation, preprint available at www-math.mit.edu/∼mhedden/.*

[MelMor] P. Melvin and H. Morton. *Fibred Knots of Genus 2 Formed by Plumbing Hopf Bands*, J. London Math. Soc., *s2-34(1)*, (1986), 159–168.

[Mil] J. Milnor. *Singular Points of Complex Hypersurfaces*, Annals of Mathematics Studies, *61*, Princeton University Press, (1968).

[NeuRud1] W. Neumann and L. Rudolph. *Difference index of vectorfields and the enhanced Milnor number*, Topology, *29*, (1990), 83–100.

[NeuRud2] W. Neumann and L. Rudolph. *Unfoldings in knot theory*, Math. Ann., *278*, (1987), 409–439.

[OzsSza1] P. S. Ozsváth and Z. Szabó. *Holomorphic disks and topological invariants for closed three-manifolds*, Annals of Math., *159(3)*, (2004), 1027–1158.

[OzsSza2] P. S. Ozsváth and Z. Szabó. *Holomorphic disks and knot invariants*, Adv. in Math., *186(1)*, (2004), 58–116.

[OzsSza3] P. S. Ozsváth and Z. Szabó. *Heegaard Floer homology and contact structures*, Duke Math. J., *129(1)*, (2005), 39–61.

[OzsSza4] P. S. Ozsváth and Z. Szabó. *Knot Floer Homology and the four-ball genus*, Geom. Topol.,*7*, (2003), 615–639.

[Ras] J. Rasmussen. *Floer homology and knot complements*, *PhD thesis, Harvard University, (2003).*

[Rud1] L. Rudolph. *Isolated critical points of mappings from $\mathbb{R}^4$ to R^2 and a natural splitting of the Milnor number of a classical fibered link. Part I*, Comm. Math. Helv., *62*, (1987), 630–645.

[Rud2] L. Rudolph. *A non-ribbon plumbing of fibered ribbon knots*, Proc. Amer. Math. Soc., *130*, (2002), 3741–3743.

[Rud3] L. Rudolph. *Knot theory of complex plane curves*, *math.GT/0411115 v2 (2004)*

[Sta] J. Stallings. *Constructions of fibred knots and links*, *Algebraic and geometric topology, in the Proceedings of the Symposium on Pure Math., Part 2, AMS (1978), 55–60.*

[ThuWin] W.P. Thurston and H.E. Winkelnkemper. *On the existence of contact forms*, Proc. Amer. Math. Soc. **52**, (1975), 345–347.

[Tor] I. Torisu. *Convex contact structures and fibered links in 3-manifolds,* Internat. Math. Res. Notices, (2000), 441–454.

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On the Khovanov and knot Floer homologies of quasi-alternating links

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Abstract. Quasi-alternating links are a natural generalization of alternating links. In this paper, we show that quasi-alternating links are "homologically thin" for both Khovanov homology and knot Floer homology. In particular, their bigraded homology groups are determined by the signature of the link, together with the Euler characteristic of the respective homology (i.e. the Jones or the Alexander polynomial). The proofs use the exact triangles relating the homology of a link with the homologies of its two resolutions at a crossing.

1. Introduction

In recent years, two homological invariants for oriented links $L \subset S^3$ have been studied extensively: Khovanov homology and knot Floer homology. Our purpose here is to calculate these invariants for the class of quasi-alternating links introduced in [19], which generalize alternating links.

The first link invariant we will consider in this paper is *Khovanov's reduced homology* ([5],[6]). This invariant takes the form of a bigraded vector space over $\mathbb{Z}/2\mathbb{Z}$, denoted $\widetilde{Kh}^{i,j}(L)$, whose Euler characteristic is the Jones polynomial in the following sense:

$$\sum_{i \in \mathbb{Z}, j \in \mathbb{Z} + \frac{l-1}{2}} (-1)^i q^j \operatorname{rank} \widetilde{Kh}^{i,j}(L) = V_L(q),$$

where l is the number of components of L. The indices i and j are called the homological and the Jones grading, respectively. (In our convention j is actually half the integral grading j from [5].) The indices appear as superscripts because Khovanov's theory is conventionally defined to be a cohomology theory. It is also useful to consider a third grading δ, described by the relation $\delta = j - i$.

Khovanov's original definition gives a theory whose Euler characteristic is the Jones polynomial multiplied by the factor $q^{1/2} + q^{-1/2}$; for the reduced theory, the Euler characteristic is the usual Jones polynomial, i.e. normalized so that it takes the value 1 on the unknot, cf. [6]. Note that $\widetilde{Kh}$ can be also be defined with integer coefficients, but then

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it depends on the choice of a component of the link. Nevertheless, $\widetilde{Kh}$ is a link invariant over $\mathbb{Z}/2\mathbb{Z}$; see [19, Section 5] or [29, Section 3].

The other homological link invariant that we consider in this paper is *knot Floer homology*. This theory was independently introduced by Szabó and the second author in [15], and by Rasmussen [27]. In its simplest form, it is a bigraded Abelian group $\widehat{HFK}_i(L, j)$ whose Euler characteristic is (up to a factor) the Alexander-Conway polynomial $\Delta_L(q)$:

$$\sum_{j \in \mathbb{Z}, i \in \mathbb{Z}+\frac{l-1}{2}} (-1)^{i+\frac{l-1}{2}} q^j \operatorname{rank} \widehat{HFK}_i(L, j) = (q^{-1/2} - q^{1/2})^{l-1} \cdot \Delta_L(q).$$

Knot Floer homology was originally defined using pseudo-holomorphic curves, but there are now also several combinatorial formulations available, cf. [10], [11], [30], [24]. The two gradings i and j are called the *Maslov* and *Alexander gradings* respectively; we also set $\delta = j - i$ as above. Knot Floer homology detects the genus of a knot [17], as well as whether a knot is fibered [25]. There exists also an improvement, called link Floer homology ([22], [23]), which detects the Thurston norm of the link complement, but that theory will not be discussed in this paper. Also, even though $\widehat{HFK}$ can be defined with integer coefficients, in this paper we will only consider it with coefficients in the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

For many classes of links (including most knots with small crossing number), the Khovanov and knot Floer homologies over $R = \mathbb{Z}$ or $\mathbb{F}$ take a particularly simple form: they are free R-modules supported in only one δ-grading. We call such links *Khovanov homologically thin (over R)*, or *Floer homologically thin (over R)*, depending on which theory we refer to. Various versions of these definitions appeared in [2], [27], [6], [28]. Further, it turns out that typically the δ-grading in which the homology groups are supported equals $-\sigma/2$, where σ is the signature of the link. When this is the case, we say that the link is (Khovanov or Floer) *homologically σ-thin*. (Floer homologically σ-thin knots were called *perfect* in [26].)

If a link L is homologically σ-thin over $R = \mathbb{Z}$ or $\mathbb{F}$ for a bigraded theory $\mathcal{H}$ (where $\mathcal{H}$ could denote either $\widetilde{Kh}$ or $\widehat{HFK}$), then $\mathcal{H}(L)$ is completely determined by the signature σ of L and the Euler characteristic $P(q)$ of $\mathcal{H}$ (the latter being either the Jones or a multiple of the Alexander polynomial). Indeed, if $P(q) = \sum a_j q^j$, we must have:

$$\mathcal{H}^{i,j}(L) \simeq \begin{cases} R^{|a_j|} & \text{if } i = j + \frac{\sigma}{2} \\ 0 & \text{otherwise.} \end{cases}$$

In the world of Khovanov homology, the fact that the vast majority (238) of the 250 prime knots with up to 10 crossings are homologically σ-thin was first observed by Bar-Natan, based on his calculations in [2]. Lee [7] showed that alternating links are Khovanov homologically σ-thin. Since 197 of the prime knots with up to 10 crossings are alternating, this provides a partial explanation for Bar-Natan's observation.

At roughly the same time, a similar story unfolded for knot Floer homology. Rasmussen [26] showed that 2-bridge knots are Floer homologically σ-thin; and this result was generalized in [16] to all alternating knots.

In this paper we generalize these results to a larger class of links, the *quasi-alternating links* of [19]. Precisely, $\mathcal{Q}$ is the smallest set of links satisfying the following properties:

- The unknot is in $\mathcal{Q}$;
- If L is a link which admits a projection with a crossing such that
  (1) both resolutions L_0 and L_1 at that crossing (as in Figure 1) are in $\mathcal{Q}$,
  (2) $\det(L) = \det(L_0) + \det(L_1)$,
  then L is in $\mathcal{Q}$.

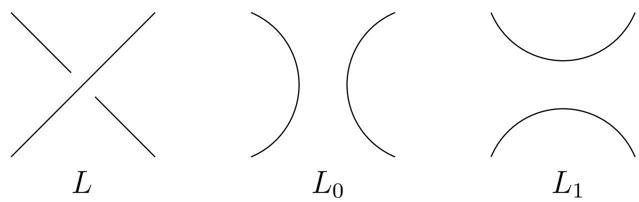

FIGURE 1. The links in the unoriented skein relation.

The elements of $\mathcal{Q}$ are called quasi-alternating links. It it is easy to see (cf. [19, Lemma 3.2]) that alternating links are quasi-alternating.

In this paper we prove the following:

Theorem 1.1. *Quasi-alternating links are Khovanov homologically σ-thin (over $\mathbb{Z}$).*

Theorem 1.2. *Quasi-alternating links are Floer homologically σ-thin (over $\mathbb{Z}/2\mathbb{Z}$).*

For knots with up to nine crossings, Theorem 1.1 and Theorem 1.2 provide an almost complete explanation for the prevalence of homological σ-thinness (over the respective coefficient ring). Indeed, among the 85 prime knots with up to nine crossings, only two (8_{19} and 9_{42}) are not Khovanov homologically σ-thin, and these are also the only ones which are not Floer homologically σ-thin. By the results of [19], [9] and [1], 82 of the 83 remaining knots are quasi-alternating. (Among them, 74 are alternating.) This leaves only the knot 9_{46}, which the authors do not know if it is quasi-alternating.

In general it is difficult to decide whether a larger, homologically σ-thin knot is quasi-alternating. It remains a challenge to find homologically σ-thin knots that are not quasi-alternating; 9_{46} could be the first potential example.

A few words are in order about the strategy of proof and the organization of the paper. Both Theorem 1.1 and Theorem 1.2 are consequences of the unoriented skein exact triangles satisfied by the respective theories. For Khovanov homology, this exact triangle (which relates the homology of L to that of its resolutions L_0 and L_1, cf. Figure 1), is immediate from the definition of the homology groups. The only new ingredient used in the proof of Theorem 1.1 is an observation about relating the gradings to the signature. We explain this in Section 2 of the paper. In fact, the proof of Theorem 1.1 is an adaptation of the proof of the corresponding fact for alternating links due to Lee [7].

For knot Floer homology, an unoriented skein exact triangle was described by the first author in [9]. In that paper, the maps in the triangle were ungraded. In Section 3, we show that they actually respect the δ-grading, up to a well-determined shift. This will imply Theorem 1.2. It is interesting to note that this strategy is quite different from the earlier proofs for two-bridge and alternating links, [27], [16].

We remark that Theorem 1.2 has a number of formal consequences. The full version of knot Floer homology is a graded, filtered chain complex over the polynomial algebra $\mathbb{F}[U]$. It was shown in [16, Theorem 1.4 and the remark immediately after] that for Floer homologically σ-thin knots, their full complex (up to equivalence) is determined by their Alexander polynomial and signature. Theorem 1.2 implies then that this is true for quasi-alternating knots. Furthermore, according to [20] and [21], the full knot Floer complex has enough information to determine the Heegaard Floer homology of any Dehn surgery on that knot. Thus, the Floer homologies (over $\mathbb{F}$) of Dehn surgeries on quasi-alternating knots are determined by the Alexander polynomial, the signature, and the surgery coefficient; we refer to [16], [20], [21] for the precise statements.

It is natural to expect Theorem 1.2 to hold also over $\mathbb{Z}$. Note that Theorem 1.2, combined with the universal coefficients theorem, implies that quasi-alternating links are Floer homologically σ-thin over $\mathbb{Q}$.

2. The exact triangle for Khovanov homology

2.1. The Gordon-Litherland formula

Let us review the definition of the Goeritz matrix, as well as the Gordon-Litherland formula for the signature, following [3].

Consider an oriented link L in S^3 with a regular, planar projection, and let D be the corresponding planar diagram. The complement of the projection in $\mathbb{R}^2$ has a number of connected components, which we call *regions*. We color them in black and white in checkerboard fashion. Let $R_0, R_1, \dots, R_n$ be the white regions. Assume that each crossing is incident to two distinct white regions. To each crossing c we assign an incidence number $\mu(c)$, as well as a type (I or II), as in Figure 2. Note that the sign of the crossing is determined by its incidence number and type.

Set

$$\mu(D) = \sum_{c \text{ of type II}} \mu(c).$$

The Goeritz matrix $G = G(D)$ of the diagram D is defined as follows. For any $i, j \in \{0, 1, \dots, n\}$ with $i \neq j$, let

$$g_{ij} = -\sum_{c \in \bar{R}_i \cap \bar{R}_j} \mu(c).$$

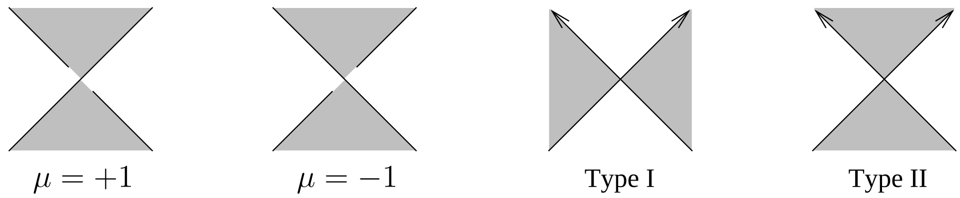

FIGURE 2. Incidence numbers and types of crossings.

Set also

$$g_{ii} = -\sum_{i \neq j} g_{ij}.$$

Then G is the $n \times n$ symmetric matrix with entries g_{ij}, for $i, j \in \{1, \ldots, n\}$.

Gordon and Litherland showed that the signature of L is given by the formula

$$\sigma(L) = \text{signature}(G) - \mu(D). \tag{1}$$

(We use the convention that the signature of the right-handed trefoil is -2.) Also, the determinant $\det(L)$ of a link L can be defined as the non-negative integer

$$\det(L) = |\det(G)|.$$

2.2. The signature of resolutions.

Let $L \subset S^3$ be an oriented link with a fixed planar projection as before. Fix now a crossing c_0 in the corresponding planar diagram. If the crossing is positive (resp. negative), we set $L_+ = L$ (resp. $L_- = L$) and let L_- (resp. L_+) be the link obtained form L by changing the sign of the crossing. Further, we denote by L_v and L_h the oriented and unoriented resolutions of L at that crossing, cf. Figure 3. (We choose an arbitrary orientation for L_h.) To make the connection with Figure 1, note that if $L = L_+$, then $L_0 = L_v$ and $L_1 = L_h$, while if $L = L_-$, then $L_0 = L_h$ and $L_1 = L_v$.

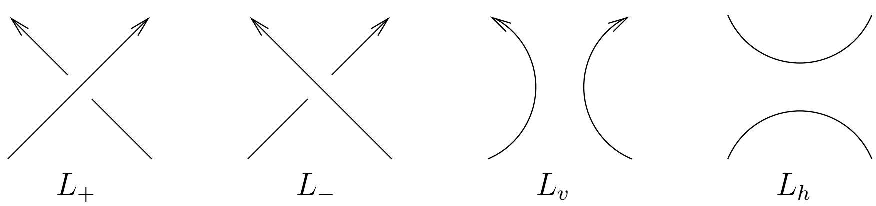

FIGURE 3. Two possible crossings and their resolutions.

Denote by D_+, D_v, D_h the planar diagrams of L_+, L_v, L_h, respectively, differing from each other only at the chosen crossing c_0.

The first equality in the lemma below (without the sign) is due to Murasugi [12]; the second is also inspired by a result of Murasugi from [13].

Lemma 2.1. *Suppose that* $\det(L_v), \det(L_h) > 0$ *and* $\det(L_+) = \det(L_v) + \det(L_h)$. *Then:*

$$\sigma(L_v) - \sigma(L_+) = 1$$

and

$$\sigma(L_h) - \sigma(L_+) = e,$$

where e *denotes the difference between the number of negative crossings in* D_h *and the number of such crossings in* D_+.

Proof. Construct the Goeritz matrices $G_+ = G(D_+), G_v = G(D_v)$ and $G_h = G(D_h)$ in such a way that c_0 is of Type I (and incidence number -1) in D_+, and the white region R_0 (the one not appearing in the Goeritz matrix) is as in Figure 4.

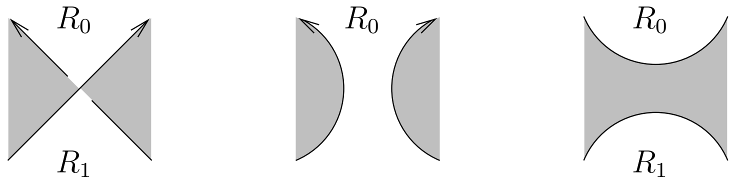

FIGURE 4. Coloring convention at the chosen crossing.

Observe now that G_+ and G_h are bordered matrices of G_v. More precisely, if G_v is an $n \times n$ symmetric matrix, then there exists $a \in \mathbb{R}$ and $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ such that

$$G_+ = \begin{pmatrix} a & v \\ v^T & G_v \end{pmatrix}; \quad G_h = \begin{pmatrix} a+1 & v \\ v^T & G_v \end{pmatrix}.$$

Without loss of generality (after an orthonormal change of basis), we can assume that G_v is diagonal, with diagonal entries $\alpha_1, \ldots, \alpha_n$. Note that these are nonzero because $\det(L_v) = |\det(G_v)| \neq 0$.

The bilinear form associated to G_+ can be written as

$$aX^2 + 2\sum_{i=1}^{n} v_i X X_i + \sum_{i=1}^{n} \alpha_i X_i^2,$$

or

$$\Big(a - \sum_{i=1}^{n} \frac{v_i^2}{\alpha_i}\Big) X^2 + \sum_{i=1}^{n} \alpha_i \Big(X_i + \frac{v_i}{\alpha_i} X\Big)^2.$$

A similar formula holds for the form of G_h, but with a replaced by $a+1$.

If we set
$$\beta = a - \sum_{i=1}^{n} \frac{v_i^2}{\alpha_i},$$
then
$$\det(G_+) = \beta \cdot \det(G_v), \quad \det(G_h) = (\beta+1)\cdot \det(G_v).$$
By the condition on the determinants in the hypothesis, $|\beta| = |\beta+1|+1$, so we must have $\beta < -1$. Therefore, when we diagonalize the bilinear forms, for G_+ (resp. G_h) we get one additional negative coefficient (β, resp. $\beta+1$) as compared to G_v. Thus,
$$\text{signature}(G_+) = \text{signature}(G_h) = \text{signature}(G_v) - 1. \tag{2}$$
Since c_0 is of Type I, we also have $\mu(D_+) = \mu(D_v)$. Together with the Gordon-Litherland formula (1), these identities imply
$$\sigma(L_+) = \sigma(L_v) - 1.$$
Next, observe that when we change the direction of an arc at a crossing, both the sign and the type of the crossing are reversed, but the incidence number remains the same. If we denote by $k(\mu, t)$ the number of crossings of incidence number $\mu \in \{\pm 1\}$ and type $t \in \{I, II\}$ in D_+ (excluding c_0) which change type (and sign) in D_h, then
$$\mu(D_h) - \mu(D_+) = k(+1, I) - k(-1, I) - k(+1, II) + k(-1, II).$$
This equals
$$-\big(k(-1,I) + k(+1,II)\big) + \big(k(+1,I) + k(-1,II)\big) = -e.$$
Using (1) and (2) again, we get
$$\sigma(L_+) = \sigma(L_h) + e,$$
as desired. □

2.3. An unoriented skein exact triangle.

The following proposition is a simple consequence of the definition of Khovanov cohomology. It is implicit in [5], and also appeared in Viro's work [31]. The statement below, with the precise gradings, is taken from Rasmussen's review [28, Proposition 4.2]. It is written there in terms of Khovanov's unreduced homology, but it works just as well for the reduced version $\widetilde{Kh}$, which we use in this paper. We work over $\mathbb{Z}$, so to define the reduced homology we need to mark a component for each link appearing in the triangle; we do this by marking the same point on their diagrams, away from the crossing where the links differ.

Proposition 2.2. *(Khovanov, Viro, Rasmussen) There are long exact sequences*
$$\cdots \to \widetilde{Kh}^{i-e-1, j-\frac{3e}{2}-1}(L_h) \to \widetilde{Kh}^{i,j}(L_+) \to \widetilde{Kh}^{i, j-\frac{1}{2}}(L_v) \to \widetilde{Kh}^{i-e, j-\frac{3e}{2}-1}(L_h) \to \cdots$$
and
$$\cdots \to \widetilde{Kh}^{i, j+\frac{1}{2}}(L_v) \to \widetilde{Kh}^{i,j}(L_-) \to \widetilde{Kh}^{i-e+1, j-\frac{3e}{2}+1}(L_h) \to \widetilde{Kh}^{i+1, j+\frac{1}{2}}(L_v) \to \cdots$$

where e is as in the statement of Lemma 2.1.

If we forget about i and j and just keep the grading $\delta = j - i$, the two triangles become

$$\cdots \to \widetilde{Kh}^{*-\frac{e}{2}}(L_h) \to \widetilde{Kh}^{*}(L_+) \to \widetilde{Kh}^{*-\frac{1}{2}}(L_v) \to \widetilde{Kh}^{*-\frac{e}{2}-1}(L_h) \to \cdots \tag{3}$$

and

$$\cdots \to \widetilde{Kh}^{*+\frac{1}{2}}(L_v) \to \widetilde{Kh}^{*}(L_-) \to \widetilde{Kh}^{*-\frac{e}{2}}(L_h) \to \widetilde{Kh}^{*-\frac{1}{2}}(L_v) \to \cdots \tag{4}$$

Proposition 2.3. *Let L be a link and L_0, L_1 its two resolutions at a crossing as in Figure 1. Assume that* $\det(L_0), \det(L_1) > 0$ *and* $\det(L) = \det(L_0) + \det(L_1)$. *Then there is an exact triangle:*

$$\cdots \to \widetilde{Kh}^{*-\frac{\sigma(L_1)}{2}}(L_1) \to \widetilde{Kh}^{*-\frac{\sigma(L)}{2}}(L) \to \widetilde{Kh}^{*-\frac{\sigma(L_0)}{2}}(L_0) \to \widetilde{Kh}^{*-\frac{\sigma(L_1)}{2}-1}(L_1) \to \cdots$$

Proof. When the given crossing in L is positive, this is a re-writing of the triangle (3), taking into account the result of Lemma 2.1. Note that when following three consecutive maps in the triangle the grading decreases by one; thus, the grading change for the map between the homologies of the two resolutions is determined by the grading change for the other two maps.

The case when the crossing is negative is similar. $\square$

Proof of Theorem 1.1. Note that any quasi-alternating link has nonzero determinant; this follows easily from the definition. The desired result is then a consequence of Proposition 2.3: the unknot is homologically σ-thin and, because of the exact triangle, if L_0 and L_1 are homologically σ-thin, then so is L. $\square$

3. The exact triangle for knot Floer homology

In this section we assume that the reader is familiar with the basics of knot Floer homology (including the version with several basepoints), cf. [15], [27], [22], [10]. Throughout this section we will work with coefficients in the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

3.1. Heegaard diagrams and periodic domains

We start with a few generalities about periodic domains in Heegaard diagrams. Our discussion is very similar to the ones in [18, Section 2.4] and [22, Section 3.4], except that here we do not ask for the periodic domains to avoid any basepoints.

Let Σ be a Riemann surface of genus g. A collection $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ of disjoint, simple closed curves on Σ is called *good* if the span S_α of the classes $[\alpha_i]$ in $H_1(\Sigma; \mathbb{Z})$ is g-dimensional. If $\boldsymbol{\alpha}$ is such a collection, we view (the closures of) the components of $\Sigma - (\cup \alpha_i)$ as two-chains on Σ and denote by Π_α their span. Note that Π_α is a free Abelian group of rank $m = n - g + 1$.

A *Heegaard diagram* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ consists of a Riemann surface Σ together with two good collections of curves $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$. (A Heegaard diagram describes a 3-manifold Y; see for example [22, Section 3.1].) We define a *periodic domain* in the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ to be a two-chain on Σ that is a linear combination of the components of $\Sigma - (\cup\alpha_i) - (\cup\beta_i)$, and with the property that its boundary is a linear combination of the alpha and beta curves. (This is a slight modification of [18, Definition 2.14].) The group of periodic domains is denoted $\Pi_{\alpha,\beta}$. Let also $S_{\alpha,\beta} = S_\alpha + S_\beta$ be the span of all the alpha and beta curves in $H_1(\Sigma;\mathbb{Z})$.

Lemma 3.1. *The group $\Pi_{\alpha,\beta}$ of periodic domains is free Abelian of rank equal to $2n+1-\operatorname{rank}(S_{\alpha,\beta})$.*

Proof. There is a map

$$\psi_{\alpha,\beta} : \mathbb{Z}^{2n} \to S_{\alpha,\beta}$$

taking the first n standard generators of $\mathbb{Z}^{2n}$ to the classes $[\alpha_i]$, $i = 1,\dots,n$, and the remaining n standard generators to the classes $[\beta_i]$, $i = 1,\dots,n$. There is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Pi_{\alpha,\beta} \longrightarrow \ker(\psi_{\alpha,\beta}) \longrightarrow 0. \tag{5}$$

Indeed, the map $\Pi_{\alpha,\beta} \to \ker(\psi_{\alpha,\beta})$ takes a periodic domain $\mathcal{D}$ to the coefficients of the alpha and beta curves appearing in $\partial\mathcal{D}$. It is surjective, and its kernel is generated by the Heegaard surface Σ itself.

The conclusion follows immediately from the short exact sequence. □

Note that we can view Π_α and Π_β as subgroups of $\Pi_{\alpha,\beta}$. Their intersection is generated by the two-chain Σ. Therefore,

$$\operatorname{rank}(\Pi_\alpha + \Pi_\beta) = 2n - 1.$$

More precisely, if we denote by $S_\alpha \oplus S_\beta \cong \mathbb{Z}^{2g}$ the exterior direct sum, there is a short exact sequence analogous to (5):

$$0 \to \mathbb{Z} \longrightarrow \Pi_\alpha + \Pi_\beta \longrightarrow \ker(\mathbb{Z}^{2n} \to S_\alpha \oplus S_\beta) \to 0. \tag{6}$$

Corollary 3.2. *If $S_{\alpha,\beta} = H_1(\Sigma;\mathbb{Z})$, then $\Pi_{\alpha,\beta} = \Pi_\alpha + \Pi_\beta$.*

Proof. The exact sequences (5) and (6) fit into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"'] & \Pi_\alpha + \Pi_\beta \arrow[r] \arrow[d] & \ker(\mathbb{Z}^{2n} \to S_\alpha \oplus S_\beta) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \Pi_{\alpha,\beta} \arrow[r] & \ker(\mathbb{Z}^{2n} \to S_{\alpha,\beta}) \arrow[r] & 0
\end{tikzcd}$$

To show that the middle vertical arrow is an isomorphism it suffices to show that the right vertical arrow is. The map $\psi_{\alpha,\beta} : \mathbb{Z}^{2n} \to S_{\alpha,\beta}$ factors through $S_\alpha \oplus S_\beta$. Consider the sequence of maps

$$\mathbb{Z}^{2g} \cong S_\alpha \oplus S_\beta \longrightarrow S_{\alpha,\beta} \hookrightarrow H_1(\Sigma;\mathbb{Z}) \cong \mathbb{Z}^{2g}.$$

The hypothesis says that the last inclusion is an isomorphism, which means that the composition is surjective. Since its domain and target are both $\mathbb{Z}^{2g}$, the map must be an isomorphism. This shows that $S_\alpha \oplus S_\beta \longrightarrow S_{\alpha,\beta}$ is an isomorphism as well. □

Finally, a *triple Heegaard diagram* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ consists of a Riemann surface Σ together with three good collections of curves $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$, and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_n)$. A *triply periodic domain* is then a two-chain on Σ that is a linear combination of the components of $\Sigma - (\cup \alpha_i) - (\cup \beta_i) - (\cup \gamma_i)$, and with the property that its boundary is a linear combination of the alpha, beta, and gamma curves.

The group of triply periodic domains is denoted $\Pi_{\alpha,\beta,\gamma}$. Set $S_{\alpha,\beta,\gamma} = S_\alpha + S_\beta + S_\gamma$ in $H_1(\Sigma; \mathbb{Z})$. A straightforward analog of Lemma 3.1 then says that $\Pi_{\alpha,\beta,\gamma}$ is a free Abelian group of rank equal to $3n + 1 - \operatorname{rank}(S_{\alpha,\beta,\gamma})$.

3.2. The ungraded triangle

The following theorem was proved in [9]:

Theorem 3.3. *Let L be a link in S^3, and L_0 and L_1 the two resolutions of L at a crossing, as in Figure 1. Denote by l, l_0, l_1 the number of components of the links L, L_0, and L_1, respectively, and set $m = \max\{l, l_0, l_1\}$. Then, there is an exact triangle*

$$\widehat{HFK}(L) \otimes V^{m-l} \to \widehat{HFK}(L_0) \otimes V^{m-l_0} \to \widehat{HFK}(L_1) \otimes V^{m-l_1} \to \widehat{HFK}(L) \otimes V^{m-l},$$

where V denotes a two-dimensional vector space over $\mathbb{F}$.

Our goal will be to study how the maps in the exact triangle behave with respect to the δ-grading. In order to do this, we recall how the maps were constructed in [9].

The starting point is a special Heegaard diagram which we associate to a regular, connected, planar projection D of the link L. (This is a suitable stabilization of the diagram considered in [16].) We assume that one of the crossings in D is c_0, such that the two resolutions at c_0 are diagrams D_0 and D_1 for L_0 and L_1, respectively. If D has k crossings, then it splits the plane into $k+2$ regions. Let A_0, A_1, A_2, A_3 be the regions near c_0 in clockwise order, as in Figure 5, and e the edge separating A_0 from A_1. We can assume that A_0 is the unbounded region in $\mathbb{R}^2 - D$. Denote the remaining regions by $A_4, \dots, A_{k+1}$. Let p be a point on the edge e. If $m = \max\{l, l_0, l_1\}$ is as in the statement of Theorem 3.3, then we can choose $p_1, \dots, p_{m-1}$ to be a collection of points in the plane, distinct from the crossings and such that for every component of any of the links L, L_0 and L_1, the projection of that component contains at least one of the points p_i or p.

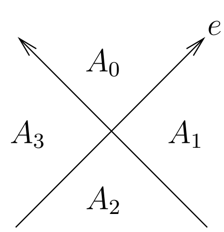

FIGURE 5. The regions near the crossing c_0. Since c_0 can be either negative or positive, we have not marked which strand is the overpass.

We denote by Σ the boundary of a regular neighborhood of D in S^3, a surface of genus $g = k + 1$. To every region A_r $(r > 0)$ we associate a curve α_r on Σ, following the boundary of A_r. To each crossing c in D we associate a curve β_c on Σ as indicated in Figure 6. In addition, we introduce an extra curve β_e which is the meridian of the knot, supported in a neighborhood of the distinguished edge e. We also mark the surface Σ with two basepoints, one on each side of β_e, as shown on the left side of Figure 7.

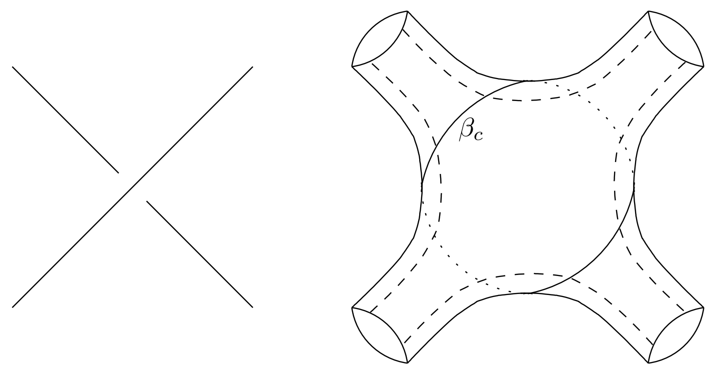

FIGURE 6. Piece of the Heegaard surface Σ associated to a crossing c. It contains four (or fewer) bits of alpha curves, shown in dashed lines, and one beta curve β_c.

Furthermore, for every edge e_i of D containing one of the points p_i, $i = 1, \ldots, m-1$, we introduce a ladybug, i.e. an additional pair of alpha-beta curves on Σ, as well as an additional pair of basepoints. This type of configuration is shown on the right side of Figure 7.

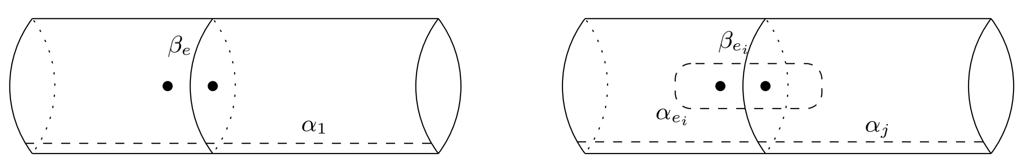

FIGURE 7. A neighborhood of the distinguished edge e (left) and a ladybug around some edge e_i marked by p_i (right).

The surface Σ, together with the collections of alpha curves, beta curves and basepoints, forms a multi-pointed Heegaard diagram for S^3 compatible with L, in the sense of [10, Definition 2.1]. We denote the alpha and the beta curves in the diagram by α_i, β_i with $i = 1, \ldots, n$, where $n = g + m - 1$. We reserve the index n for the beta curve $\beta = \beta_n$

associated to the crossing c_0. Also, we let $\widehat{\Sigma}$ denote the complement of the basepoints in the surface Σ.

We can construct similar Heegaard diagrams compatible with L_0 and L_1 as follows. The surface Σ, the alpha curves and the basepoints remain the same. However, for L_0 we replace the beta curves by gamma curves $\gamma_i,\ i = 1, \ldots, n$, while for L_1 we use delta curves $\delta_i,\ i = 1, \ldots, n$. For $i < n$, the curves γ_i and δ_i are small isotopic translates of β_i, such that they intersect β_i in two points, and they also intersect each other in two points. For $i = n$, we draw the curves $\gamma = \gamma_n$ and $\delta = \delta_n$ as in Figure 8; see also Figure 9, where the following intersection points are labelled:

$$\beta \cap \gamma = \{A, U\}, \quad \gamma \cap \delta = \{B, V\}, \quad \delta \cap \beta = \{C, W\}.$$

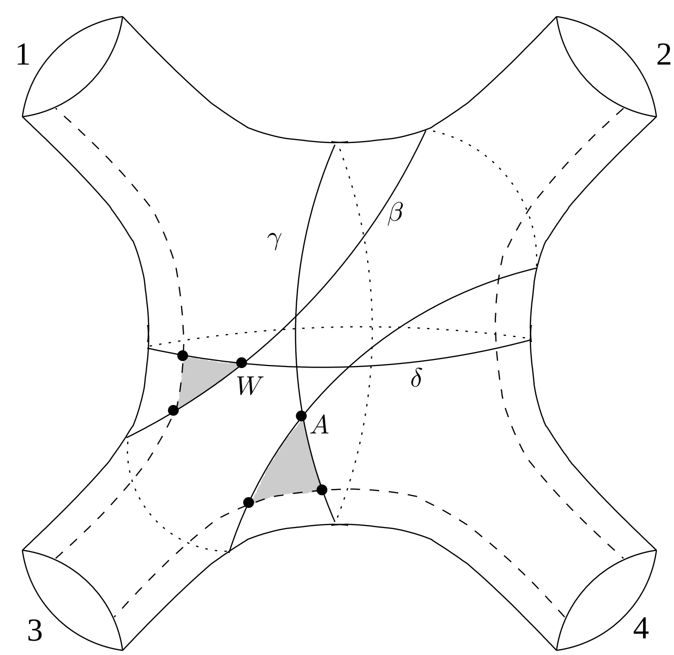

FIGURE 8. Piece of Σ near the crossing c_0. There are three bits of alpha curves, shown dashed. This piece is joined to the rest of the diagram by four tubes, which we mark by the numbers 1,2,3,4.

For the purpose of defining Floer homology, we need to ensure that the Heegaard diagrams for L, L_0 and L_1 constructed above are admissible in the sense of [22, Definition 3.5]. We achieve admissibility by stretching one tip of the alpha curve of each ladybug, and bringing it close to the basepoints associated to the distinguished edge e. It is easy

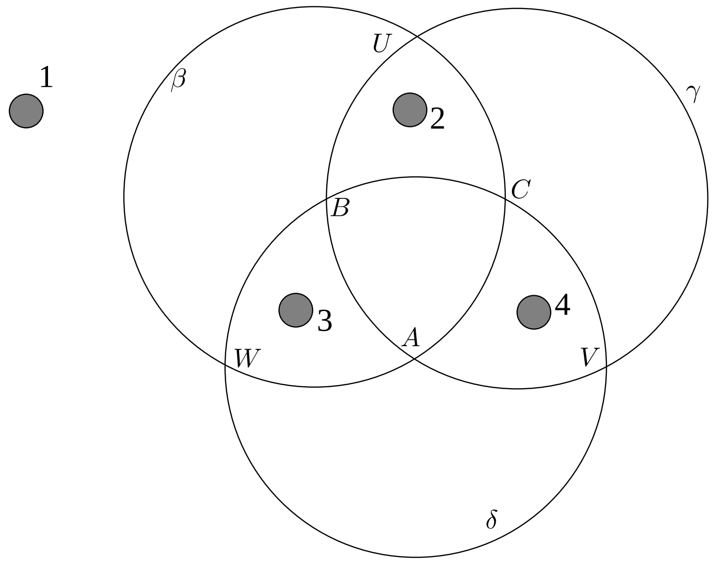

FIGURE 9. A different view of Figure 8. The four gray disks correspond to the four tubes from Figure 8, and are marked accordingly.

to see that the result is admissible; see Figure 10 for an example. In that figure, to get the diagrams for L_0 and L_1, which are both the unknot, we replace $\beta = \beta_4$ by curves γ and δ, respectively, as in Figure 8.

Now consider the tori

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_n, \quad \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_n,$$
$$\mathbb{T}_\gamma = \gamma_1 \times \cdots \times \gamma_n, \quad \mathbb{T}_\delta = \delta_1 \times \cdots \times \delta_n,$$

which we view as totally real submanifolds of the symmetric product $\mathrm{Sym}^n(\widehat{\Sigma})$. The Floer complex $CF(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ is the vector space freely generated by the intersection points between $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, and endowed with the differential

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \hat{\pi}_2(\mathbf{x},\mathbf{y}) \big| \mu(\phi)=1\}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) \mathbf{y}. \tag{7}$$

Here $\hat{\pi}_2(\mathbf{x}, \mathbf{y})$ denotes the space of homology classes of Whitney disks connecting $\mathbf{x}$ to $\mathbf{y}$ in $\mathrm{Sym}^n(\widehat{\Sigma})$, $\mathcal{M}(\phi)$ denotes the moduli space of pseudo-holomorphic representatives of ϕ (with respect to a suitable almost complex structure as in [18]), $\mu(\phi)$ denotes its formal dimension (Maslov index), and the $\#$ sign denotes the mod 2 count of points in the (zero-dimensional) moduli space. (We will henceforth use μ to denote Maslov index, rather than the incidence number, as in Section 2.)

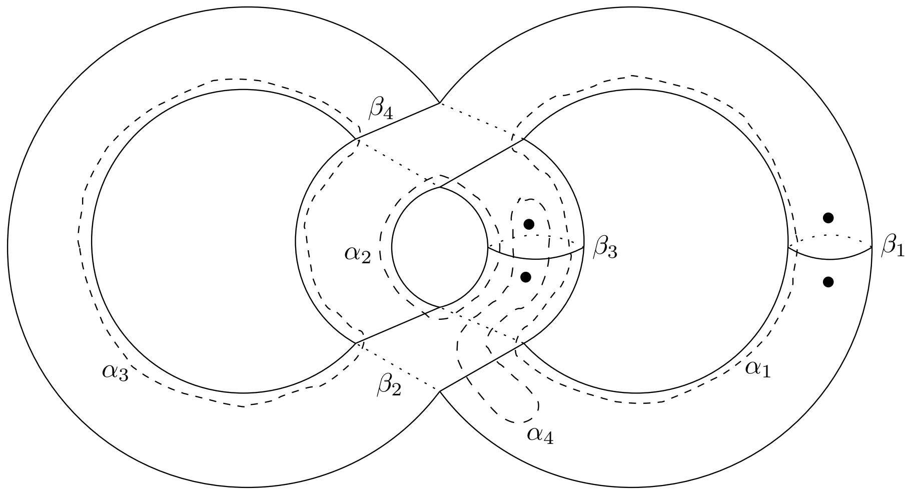

Figure 10. A Heegaard diagram compatible with the Hopf link L, with $g = 3$, $m = 2$ and $n = 4$. The beta curves β_2 and $\beta = \beta_4$ are associated to the two crossings, β_1 to the distinguished edge, and β_3 is part of a ladybug. There are three alpha curves associated to planar bounded regions and one, α_4, which is part of a ladybug. One tip of α_4 is stretched to achieve admissibility.

The homology of $CF(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ is the Floer homology $HF(\mathbb{T}_\alpha, \mathbb{T}_\beta)$. Up to a factor, this is the knot Floer homology of L:

$$HF(\mathbb{T}_\alpha, \mathbb{T}_\beta) \cong \widehat{HFK}(L) \otimes V^{m-l},$$

where V is a two-dimensional vector space as in Theorem 3.3.

We can similarly take the Floer homology of $\mathbb{T}_\alpha$ and $\mathbb{T}_\gamma$, or $\mathbb{T}_\alpha$ and $\mathbb{T}_\delta$, and obtain

$$HF(\mathbb{T}_\alpha, \mathbb{T}_\gamma) \cong \widehat{HFK}(L_0) \otimes V^{m-l_0},$$

$$HF(\mathbb{T}_\alpha, \mathbb{T}_\delta) \cong \widehat{HFK}(L_1) \otimes V^{m-l_1}.$$

Therefore, the exact triangle from Theorem 3.3 can be written as

$$HF(\mathbb{T}_\alpha, \mathbb{T}_\delta) \xrightarrow{(f_1)_*} HF(\mathbb{T}_\alpha, \mathbb{T}_\beta) \xrightarrow{(f_2)_*} HF(\mathbb{T}_\alpha, \mathbb{T}_\gamma) \xrightarrow{(f_3)_*} HF(\mathbb{T}_\alpha, \mathbb{T}_\delta) \tag{8}$$

The maps $(f_i)_*$ $(i = 1, 2, 3)$ from the triangle (8) are all induced by chain maps f_i between the corresponding Floer complexes. To define the maps f_i, let us first recall the definition of the usual triangle maps appearing in Floer theory. Given totally real submanifolds T_1, T_2, T_3 in a symplectic manifold (satisfying several technical conditions which will hold in our situations), there is a chain map

$$CF(T_1, T_2) \otimes CF(T_2, T_3) \to CF(T_1, T_3),$$

defined by counting pseudo-holomorphic triangles. In particular, given an intersection point $\mathbf{z} \in T_2 \cap T_3$ which is a cycle when viewed as an element of $CF(T_2, T_3)$, we have a chain map

$$F_{\mathbf{z}}(\mathbf{x}) = \sum_{\mathbf{y} \in T_1 \cap T_3} \sum_{\{\phi \in \hat{\pi}_2(\mathbf{x},\mathbf{z},\mathbf{y}) \big| \mu(\phi)=0\}} \#\left(\mathcal{M}(\phi)\right) \mathbf{y}.$$

Here $\hat{\pi}_2(\mathbf{x}, \mathbf{z}, \mathbf{y})$ denotes the space of homology classes of triangles with edges on T_1, T_2, T_3 and vertices $\mathbf{x}, \mathbf{z}$ and $\mathbf{y}$, respectively (in clockwise order), μ is the Maslov index, and $\#\left(\mathcal{M}(\phi)\right)$ the number of their pseudo-holomorphic representatives.

Going back to our set-up, whenever we have two isotopic curves η and η' on the surface Σ such that they intersect in exactly two points, we will denote by $M_{\eta\eta'} \in \eta \cap \eta'$ the top degree generator of $CF(\eta, \eta')$. Given one of the intersection points in Figure 9, for example $A \in \beta \cap \gamma$, we obtain a corresponding intersection point in $\mathbb{T}_\beta \cap \mathbb{T}_\gamma$ by adjoining to A the top degree intersection points $M_{\beta_i\gamma_i} \in \beta_i \cap \gamma_i$. We denote the resulting generators by the respective lowercase letters in bold:

$$\begin{aligned}
\mathbf{a} &= M_{\beta_1\gamma_1} \times M_{\beta_2\gamma_2} \times \cdots \times M_{\beta_{n-1}\gamma_{n-1}} \times A \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma;\\
\mathbf{b} &= M_{\gamma_1\delta_1} \times M_{\gamma_2\delta_2} \times \cdots \times M_{\gamma_{n-1}\delta_{n-1}} \times B \in \mathbb{T}_\gamma \cap \mathbb{T}_\delta;\\
\mathbf{c} &= M_{\delta_1\beta_1} \times M_{\delta_2\beta_2} \times \cdots \times M_{\delta_{n-1}\beta_{n-1}} \times C \in \mathbb{T}_\delta \cap \mathbb{T}_\beta;\\
\mathbf{u} &= M_{\beta_1\gamma_1} \times M_{\beta_2\gamma_2} \times \cdots \times M_{\beta_{n-1}\gamma_{n-1}} \times U \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma;\\
\mathbf{v} &= M_{\gamma_1\delta_1} \times M_{\gamma_2\delta_2} \times \cdots \times M_{\gamma_{n-1}\delta_{n-1}} \times V \in \mathbb{T}_\gamma \cap \mathbb{T}_\delta;\\
\mathbf{w} &= M_{\delta_1\beta_1} \times M_{\delta_2\beta_2} \times \cdots \times M_{\delta_{n-1}\beta_{n-1}} \times W \in \mathbb{T}_\delta \cap \mathbb{T}_\beta.
\end{aligned}$$

The chain maps f_i giving rise to (8) are then defined to be the sums

$$\begin{aligned}
f_1 &= F_{\mathbf{c}} + F_{\mathbf{w}} : CF(\mathbb{T}_\alpha, \mathbb{T}_\delta) \to CF(\mathbb{T}_\alpha, \mathbb{T}_\beta);\\
f_2 &= F_{\mathbf{a}} + F_{\mathbf{u}} : CF(\mathbb{T}_\alpha, \mathbb{T}_\beta) \to CF(\mathbb{T}_\alpha, \mathbb{T}_\gamma);\\
f_3 &= F_{\mathbf{b}} + F_{\mathbf{v}} : CF(\mathbb{T}_\alpha, \mathbb{T}_\gamma) \to CF(\mathbb{T}_\alpha, \mathbb{T}_\delta).
\end{aligned}$$

3.3. Periodic domains

Let us apply the discussion in Section 3.1 to the setting of Section 3.2.

Note that $(\Sigma; \boldsymbol{\alpha}, \boldsymbol{\beta})$, for example, is a Heegaard diagram for S^3, hence the alpha and the beta curves span all of $H_1(\Sigma; \mathbb{Z})$. Applying Corollary 3.2 we deduce that

$$\Pi_{\alpha,\beta} = \Pi_\alpha + \Pi_\beta. \tag{9}$$

Similarly, we have $\Pi_{\alpha,\gamma} = \Pi_\alpha + \Pi_\gamma$ and $\Pi_{\alpha,\delta} = \Pi_\alpha + \Pi_\delta$.

The situation for $\Pi_{\beta,\gamma}$ is different. Before analyzing it, let us first understand the components of $\Sigma - (\cup\beta_i)$, which span Π_β, in detail. Their number is m, which equals either l or $l+1$, according to whether the two strands of L meeting at c are on different link components, or on the same link component. Let $K_1, \dots, K_l$ be the connected components of L, such that K_l is the one containing the edge e. If $m = l$, then each K_i corresponds to a unique component $\mathcal{D}_i^\beta$ of $\Sigma - (\cup\beta_i)$, which lies in a neighborhood of K_i (when Σ is viewed as the boundary of a neighborhood of L). If $m = l + 1$, then for $i < l$, each

K_i corresponds again to some $\mathcal{D}_i^\beta$, but in the neighborhood of K_l there are now two components of $\Sigma - (\cup\beta_i)$, which we denote by $\mathcal{D}_l^\beta$ and $\mathcal{D}_{l+1}^\beta$, such that $\mathcal{D}_l^\beta$ is the one whose boundary contains the curve $\beta = \beta_n$.

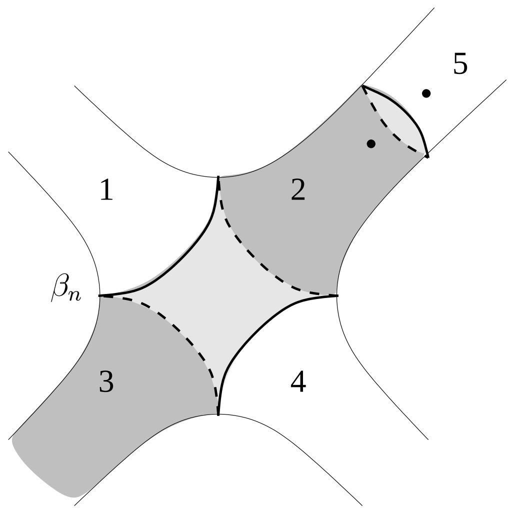

FIGURE 11. We illustrate here Equation (10). The component $\mathcal{D}_l^\beta$, which gives a homological relation between β_n and other β-curves, is shaded. There are two cases: when $m = \ell$, the region labelled here by 5 is included in $\mathcal{D}_l^\beta$. Otherwise, when $m = \ell + 1$, $\mathcal{D}_l^\beta$ terminates in a different meridional β-circle. In either case, the boundary of $\mathcal{D}_l^\beta$ consists of β-circles, and it contains β_n with multiplicity one.

Note that, regardless of whether $m = l$ or $m = l + 1$, the component $\mathcal{D}_l^\beta$ contains the curve β_n with multiplicity ± 1 (see Figure 11). This means that the class $[\beta_n] \in S_\beta \subset H_1(\Sigma; \mathbb{Z})$ is in the span of the other beta curves. In other words,

$$S_\beta = \text{ Span } (\beta_1, \ldots, \beta_{n-1}). \tag{10}$$

Similar remarks apply to $\Sigma - (\cup\gamma_i)$ and $\Sigma - (\cup\delta_i)$. Their components are denoted $\mathcal{D}_i^\gamma$ and $\mathcal{D}_i^\delta$, respectively, for $i = 1, \ldots, m$. Recall that for each $i = 1, \ldots, n-1$, the curves β_i, γ_i and δ_i are isotopic. Therefore, Equation (10), together with its analogs for the beta and gamma curves, implies that

$$S_\beta = S_\gamma = S_\delta. \tag{11}$$

For each $j = 1, \ldots, n-1$, the curves β_j and γ_j are separated by two thin bigons in Σ. The difference of these bigons is a periodic domain $\mathcal{D}_j^{\beta,\gamma}$, with boundary $\beta_j - \gamma_j$. Equation (11) implies that $\text{rank}(S_{\beta,\gamma}) = \text{rank}(S_\beta) = g$, so from Lemma 3.1 we deduce that $\operatorname{rank}(\Pi_{\beta,\gamma}) = 2n + 1 - g = n + m$. In fact, it is not hard to check that the following is true:

Lemma 3.4. *The domains* $\mathcal{D}_i^{\beta}, \mathcal{D}_i^{\gamma}$ $(i = 1, \dots, m)$ *and* $\mathcal{D}_j^{\beta,\gamma}$ $(j = 1, \dots, n-1)$ *span the group* $\Pi_{\beta,\gamma}$.

Note that we gave a set of $2m+n-1$ generators for the group $\Pi_{\beta,\gamma}$ of rank $n+m$. There are indeed $m-1$ independent relations between these generators, namely for each of the $m-1$ components K_i of L (or L_0) not containing either of the strands intersecting at c, the difference $\mathcal{D}_i^{\beta} - \mathcal{D}_i^{\gamma}$ can also be written as a sum of some domains $\mathcal{D}_j^{\beta,\gamma}$ (corresponding to the crossings on K_i).

Next, let us look at the triply periodic domains with boundary on the alpha, beta, and gamma curves.

Lemma 3.5. *We have* $\Pi_{\alpha,\beta,\gamma} = \Pi_{\alpha} + \Pi_{\beta,\gamma}$.

Proof. Let $\mathcal{D}$ be any triply periodic domain in $\Pi_{\alpha,\beta,\gamma}$. If the curve γ_n appears (with nonzero multiplicity) in the boundary of $\mathcal{D}$, by the analog of (10) for gamma curves we can subtract some domain in $\Pi_{\gamma} \subset \Pi_{\beta,\gamma}$ from $\mathcal{D}$ and obtain a new domain, in which the multiple of γ_n from $\partial\mathcal{D}$ was traded for a combination of the other gamma curves $\gamma_1, \dots, \gamma_{n-1}$. Next, whenever we have some curve γ_j in the boundary $(j < n)$, we can add the corresponding domain $\mathcal{D}_j^{\beta,\gamma} \in \Pi_{\beta} \subset \Pi_{\beta,\gamma}$ to trade it for a beta curve. Thus we arrive at a domain in $\Pi_{\alpha,\beta}$ and the conclusion follows from Equation (9). $\square$

Note that Lemma 3.4 has straightforward analogs about the structure of the groups $\Pi_{\gamma,\delta}$ and $\Pi_{\delta,\beta}$. Similarly, Lemma 3.5 has straightforward analogs about the structure of the groups $\Pi_{\alpha,\gamma,\delta}$ and $\Pi_{\alpha,\delta,\beta}$.

3.4. The relative δ-grading

Pick $\mathbf{x}, \mathbf{y} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$. Let $\pi_2(\mathbf{x}, \mathbf{y})$ be the space of homology classes of Whitney disks in $\mathrm{Sym}^n(\Sigma)$ connecting $\mathbf{x}$ and $\mathbf{y}$. (Recall that $\hat{\pi}_2(\mathbf{x}, \mathbf{y})$ is the corresponding space in $\mathrm{Sym}^n(\widehat{\Sigma})$.) Since $(\Sigma, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)$ is a Heegaard diagram for S^3, we have $\pi_2(\mathbf{x}, \mathbf{y}) \neq \emptyset$ for any $\mathbf{x}$ and $\mathbf{y}$. Note that $\pi_2(\mathbf{x}, \mathbf{x})$ can be identified with the group of periodic domains $\Pi_{\alpha,\beta}$.

Every class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ has a Maslov index $\mu(\phi) \in \mathbb{Z}$. In the usual construction of knot Floer homology, the extra basepoints on the Heegaard surface Σ are of two types: half of them are denoted w_j and the other half z_j, with $j = 1, \cdots, m+1$, such that every connected component of $\Sigma - \cup\alpha_i$ or $\Sigma - \cup\beta_i$ contains exactly one of the w_j and one of the z_k. Let $W(\phi)$ and $Z(\phi)$ be the intersection numbers of ϕ with the union of all $\{w_j\} \times \mathrm{Sym}^{n-1}(\Sigma)$ and the union of all $\{z_j\} \times \mathrm{Sym}^{n-1}(\Sigma)$, respectively. Thus $\hat{\pi}_2(\mathbf{x}, \mathbf{y})$ is the space of classes ϕ with $W(\phi) = Z(\phi) = 0$.

The difference in the Maslov grading H (denoted i in the introduction) between $\mathbf{x}$ and $\mathbf{y}$ can be calculated by picking some $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ and applying the formula

$$H(\mathbf{x}) - H(\mathbf{y}) = \mu(\phi) - 2W(\phi).$$

Similarly, the difference in the Alexander grading A (denoted j in the introduction) is

$$A(\mathbf{x}) - A(\mathbf{y}) = Z(\phi) - W(\phi).$$

Setting $P(\phi) = Z(\phi) + W(\phi)$, the difference in the grading $\delta = A - H$ is then

$$\delta(\mathbf{x}) - \delta(\mathbf{y}) = P(\phi) - \mu(\phi).$$

Therefore, if we limit ourselves to considering the δ grading, there is no difference between the two types of basepoints. This explains why we have not distinguished between them in Section 3.2, and we will not distinguish between them from now on either.

Observe that the relative δ grading is well-defined, i.e. we have $\mu(\phi) - P(\phi) = \mu(\phi') - P(\phi')$ for any $\phi, \phi' \in \pi_2(\mathbf{x}, \mathbf{y})$. Indeed, because μ and P are additive under concatenation, it suffices to prove that $\mu(\phi) - P(\phi) = 0$ for any $\phi \in \pi_2(\mathbf{x}, \mathbf{x}) = \Pi_{\alpha\beta}$. By Equation (9), the group $\Pi_{\alpha\beta}$ is generated by the connected components of $\Sigma - \cup\alpha_i$ and $\Sigma - \cup\beta_i$. Each such component has $\mu(\phi) = P(\phi) = 2$, so the relative δ grading is well-defined.

Lemma 3.6. *The chain maps f_1, f_2, f_3 that induce the triangle* (8) *preserve the relative δ grading.*

Proof. First, observe that a triangle map such as $F_{\mathbf{a}} : CF(\mathbb{T}_\alpha, \mathbb{T}_\beta) \to CF(\mathbb{T}_\alpha, \mathbb{T}_\gamma)$ preserves the relative δ grading. In other words, we need to show that adding a triply periodic domain $\mathcal{D} \in \Pi_{\alpha,\beta,\gamma}$ to a class $\phi \in \hat{\pi}_2(\mathbf{x}, \mathbf{a}, \mathbf{y})$ does not change the quantity $\mu(\phi) - P(\phi)$. By Lemmas 3.4 and 3.5, it suffices to show that the classes of the domains $\mathcal{D}_i^\alpha, \mathcal{D}_i^\beta, \mathcal{D}_i^\gamma$ and $\mathcal{D}_j^{\beta,\gamma}$ all have $\mu = P$. Indeed, for $\mathcal{D}_i^\alpha, \mathcal{D}_i^\beta$ and $\mathcal{D}_i^\gamma$ this is the argument in the paragraph before Lemma 3.6, while for each $\mathcal{D}_j^{\beta,\gamma}$ $(j = 1, \dots, n-1)$ we have $\phi = P = 0$.

Next, in order to show that $f_2 = F_{\mathbf{a}} + F_{\mathbf{u}}$ preserves the relative δ-grading, we exhibit a class $\phi \in \pi_2(\mathbf{a}, \mathbf{u})$ with $\mu(\phi) = P(\phi)$. In Figure 9 there is a bigon relating A and U which is connected by the tube numbered 2 to the rest of the Heegaard diagram. This bigon is also shown on the left in Figure 12. Following the tube, we encounter several disks (or possibly none) bounded by beta circles as in the middle of Figure 12, until we find a disk as on the right of Figure 12. Lipshitz's formula for the Maslov index [8] says that $\mu(\phi)$ can be computed as the sum of the Euler measure $e(\phi)$ and a vertex multiplicity $n(\phi)$. (We refer to [8] for the definitions.) The punctured bigon on the left of Figure 12 contributes $-\frac{1}{2}$ to $e(\phi)$ and $\frac{1}{2}$ to $n(\phi)$, each middle disk -1 to $e(\phi)$ and 1 to $n(\phi)$, and the disk on the right 0 to $e(\phi)$ and 1 to $n(\phi)$. Thus $\mu(\phi) = P(\phi) = 1$.

The arguments for f_1 and f_3 are similar. □

3.5. The absolute δ-grading

The generators $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ are of two kinds. They all consist of n-tuples of points in Σ, one on each alpha curve and on each beta curve. If for each ladybug (consisting of a pair of curves α_i and β_i), $\mathbf{x}$ contains one of the two points in $\alpha_i \cap \beta_i$, we call the generator $\mathbf{x}$ *Kauffman*. Otherwise, it is called *non-Kauffman*. Note that, if we hadn't had to stretch the alpha curves on the ladybugs to achieve admissibility, all generators would have been Kauffman.

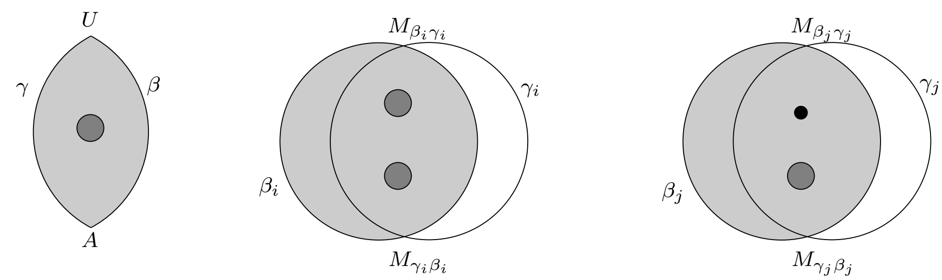

FIGURE 12. A relative homology class $\phi \in \pi_2(\mathbf{a}, \mathbf{u})$ of Maslov index 1, with one basepoint. The grey disks here denote tubes, whereas the small black dot in the rightmost picture denotes a basepoint.

Every $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ has an absolute δ-grading $\delta(\mathbf{x}) \in \frac{1}{2}\mathbb{Z}$. We will explain now a simple formula for $\delta(\mathbf{x})$ when $\mathbf{x}$ is Kauffman.

Consider the regions $A_0, A_1, A_2, \dots, A_{k+1}$ as in the second paragraph after the statement of Theorem 3.3. Each of the k crossings in D is on the boundary of four regions. A *state*, cf. [4], is an assignment which associates to each crossing one of the four incoming quadrants, such that the quadrants associated to distinct vertices are in distinct regions, and none are corners of the regions A_0 or A_1.

One can associate a monomial to each state such that as we sum all these monomials we obtain the Alexander polynomial of the link L, [4]. Therefore, if the Alexander polynomial $\Delta_L(q)$ is nonzero (or, in particular, if $\Delta_L(-1) = \det(L) \neq 0$), then there must be at least one state.

To each Kauffman generator $\mathbf{x}$ we can associate a state in an natural way: at each crossing c the corresponding beta curve intersects exactly one of the alpha curves of the neighboring regions in a point of $\mathbf{x}$, and the quadrant in that region is the one we associate to c. In [16], the Maslov and Alexander gradings of Kauffman generators are calculated in terms of their states; compare also [14].

For our purposes, it suffices to know how to compute the δ-grading. If $\mathbf{x}$ is Kauffman and c is a crossing in D, we let $\delta(\mathbf{x}, c) \in \{0, \pm 1/2\}$ be the quantity from Figure 13, chosen according to which quadrant at c appears in the state of $\mathbf{x}$. Then:

$$\delta(\mathbf{x}) = \sum_c \delta(\mathbf{x}, c). \tag{12}$$

A similar discussion applies to the diagrams D_0 and D_1 of the resolutions L_0 and L_1, respectively, except that in those cases there is no contribution from the resolved crossing c_0.

Note that the δ-grading of a Kauffman generator $\mathbf{x}$ does not depend on which of the two intersection points between the two curves of a ladybug appears in $\mathbf{x}$.

FIGURE 13. Contributions to the δ-grading.

Lemma 3.7. *Suppose that c_0 is a positive crossing in D (so that L_0 is the oriented resolution L_v) and that* $\det(L_0) \neq 0$. *Then the map $f_2 : CF(\mathbb{T}_\alpha, \mathbb{T}_\beta) \to CF(\mathbb{T}_\alpha, \mathbb{T}_\gamma)$ decreases δ-grading by* $1/2$.

Proof. By Lemma 3.6, we already know that f_2 preserves the relative δ-grading. Thus, it suffices to exhibit two generators $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ with $\delta(\mathbf{x}) - \delta(\mathbf{y}) = 1/2$, and such that there exists a holomorphic triangle of index zero in $\hat{\pi}_2(\mathbf{x}, \mathbf{a}, \mathbf{y})$.

Since $\det(L_0) \neq 0$, the diagram D_0 has at least one Kauffman generator $\mathbf{y}$. There is a corresponding Kauffman generator $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, such that each $y_i \in \gamma_i \cap \mathbf{y}$, $(i < n)$ is close to some $x_i \in \beta_i \cap \mathbf{x}$ (they are related by the isotopy between γ_i and β_i), while $x_n \in \beta \cap \mathbf{x}$ and $y_n \in \gamma \cap \mathbf{y}$ are two vertices of the shaded triangle in Figure 8 with the third vertex at A. That shaded triangle, coupled with the small triangles with vertices at $\mathbf{x}_i, \mathbf{y}_i$, and M_{β_i,γ_i} for $i = 1, \dots, n-1$, gives the desired holomorphic triangle in $\mathrm{Sym}^n(\widehat{\Sigma})$. To check that $\delta(\mathbf{x}) - \delta(\mathbf{y}) = 1/2$, note that in formula (12) the contributions to $\delta(\mathbf{x})$ and $\delta(\mathbf{y})$ from each crossing are the same, except that there is an extra contribution of $1/2$ to $\delta(\mathbf{x})$ coming from c_0. □

Lemma 3.8. *Suppose that c_0 is a positive crossing in D (so that L_1 is the unoriented resolution L_h) and that* $\det(L_1) \neq 0$. *Then the map $f_1 : CF(\mathbb{T}_\alpha, \mathbb{T}_\delta) \to CF(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ shifts δ-grading by $e/2$, where e is as in the statement of Lemma 2.1.*

Proof. By Lemma 3.6, we already know that f_1 preserves the relative δ-grading. Again, it suffices to exhibit two generators $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$ with $\delta(\mathbf{x}) - \delta(\mathbf{y}) = e/2$, and such that there exists a holomorphic triangle of index zero in $\hat{\pi}_2(\mathbf{y}, \mathbf{w}, \mathbf{x})$.

Since $\det(L_1) \neq 0$, we can pick a Kauffman generator $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$. As in the proof of Lemma 3.7, there is a corresponding Kauffman generator $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and a holomorphic triangle of index zero as desired, consisting of $n-1$ small triangles with one vertex at M_{δ_i,β_i} for $i = 1, \dots, n-1$, and the shaded triangle in Figure 8 with one vertex at W.

To check that $\delta(\mathbf{x}) - \delta(\mathbf{y}) = e/2$, let n_+ be the number of positive crossings in D (excluding c_0) which change sign in D_1. At each such crossing c, we have:

$$\delta(\mathbf{x}, c) = \delta(\mathbf{y}, c) + 1/2.$$

Let also n_- be the number of negative crossings in D which change sign in D_1. At each such crossing c, we have:

$$\delta(\mathbf{x}, c) = \delta(\mathbf{y}, c) - 1/2.$$

Therefore,

$$\delta(\mathbf{x}) - \delta(\mathbf{y}) = (n_+ - n_-)/2 = e/2.$$

$\square$

Proposition 3.9. *Let* L *be a link and* L_0, L_1 *its two resolutions at a crossing as in Figure 1. Assume that* $\det(L_0), \det(L_1) > 0$ *and* $\det(L) = \det(L_0) + \det(L_1)$*. Then, two of the three maps in the exact triangle from Theorem 3.3 behave as follows with respect to the* δ*-grading:*

$$\widehat{HFK}_{*-\frac{\sigma(L_1)}{2}}(L_1) \otimes V^{m-l_1} \to \widehat{HFK}_{*-\frac{\sigma(L)}{2}}(L) \otimes V^{m-l_0} \to \widehat{HFK}_{*-\frac{\sigma(L_0)}{2}}(L_0) \otimes V^{m-l},$$

where V *denotes a two-dimensional vector space over* $\mathbb{F}$*, in grading zero.*

Proof. When the given crossing in L is positive, this follows from (8), together with the results of Lemmas 2.1, 3.7, and 3.8. The case when the crossing is negative is similar. $\square$

Proof of Theorem 1.2. Using Proposition 3.9, we can argue in the same way as in the proof of Theorem 1.1. Note that we do not have to know the change in the absolute δ-grading under the third map $(f_3)_* : HF(\mathbb{T}_\alpha, \mathbb{T}_\gamma) \to HF(\mathbb{T}_\alpha, \mathbb{T}_\delta)$ in the exact triangle. Indeed, recall that the Euler characteristic of $\widehat{HFK}$ is (up to a factor) the Alexander polynomial, which evaluated at -1 gives the determinant of the link. If we know that L_0 and L_1 are Floer homologically σ-thin and we want to show the same for L, the fact that $\det(L) = \det(L_0) + \det(L_1)$ together with the ungraded triangle implies that

$$\text{rank } (\widehat{HFK}(L) \otimes V^{m-l}) = \text{rank } (\widehat{HFK}(L_0) \otimes V^{m-l_0}) + \text{rank } (\widehat{HFK}(L_1) \otimes V^{m-l_1}).$$

Hence $(f_3)_* = 0$, and the inductive step goes through. $\square$

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References

[1] J. Baldwin, *Quasi-alternating 3-braid links*, in preparation.

[2] D. Bar-Natan, *On Khovanov's categorification of the Jones polynomial,* Alg. Geom. Top. **2** (2002), 337-370.

[3] C. Gordon and R. Litherland, *On the signature of a link,* Invent. Math. **47** (1978), 53-69.

[4] L. H. Kauffman, *Formal knot theory*, Mathematical Notes **30**, Princeton University Press (1983).

[5] M. Khovanov, *A categorification of the Jones polynomial,* Duke Math. J. **101** (2000), no. 3, 359-426.

[6] M. Khovanov, *Patterns in knot cohomology I,* Experiment. Math. **12** (2003), no. 3, 365–374.

[7] E. S. Lee, *The support of the Khovanov's invariants for alternating knots,* preprint (2002), math/0201105.

[8] R. Lipshitz, *A cylindrical reformulation of Heegaard Floer homology,* Geom. Topol. **10** (2006), 955–1097.

[9] C. Manolescu, *An unoriented skein exact triangle for knot Floer homology,* Math. Res. Lett. **14** (2007), 839–852.

[10] C. Manolescu, P. Ozsváth and S. Sarkar, *A combinatorial description of knot Floer homology,* math/0607691, Ann. of Math., to appear.

[11] C. Manolescu, P. Ozsváth, Z. Szabó and D. Thurston, *On combinatorial link Floer homology,* Geom. Topol. **11** (2007),2339–2412.

[12] K. Murasugi,*On a certain numerical invariant of link types,* Trans. Amer. Math. Soc. **117** (1965), 387–422.
[13] K. Murasugi, *On the signature of links,* Topology **9** (1970), 283–298.
[14] P. Ozsváth, A. Stipsicz, and Z. Szabó *Floer homology and singular knots*, preprint (2007), math/0705.2661.
[15] P. Ozsváth and Z. Szabó, *Holomorphic disks and knot invariants,* Adv. Math. **186** (2004), no. 1, 58–116.
[16] P. Ozsváth and Z. Szabó, *Heegaard Floer homology and alternating knots,* Geom. Topol. **7** (2003), 225–254.
[17] P. Ozsváth and Z. Szabó, *Holomorphic disks and genus bounds,* Geom. Topol. **8** (2004), 311–334 .
[18] P. Ozsváth and Z. Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) **159** (2004), no. 3, 1027–1158.
[19] P. Ozsváth and Z. Szabó, *On the Heegaard Floer homology of branched double-covers,* Adv. Math. **194** (2005), 1-33.
[20] P. Ozsváth and Z. Szabó, *Knot Floer homology and integer surgeries,* preprint (2004), math/0410300.
[21] P. Ozsváth and Z. Szabó, *Knot Floer homology and rational surgeries,* preprint (2005), math/0504404.
[22] P. Ozsváth and Z. Szabó, *Holomorphic disks and link invariants,* preprint (2005), math/0512286.
[23] P. Ozsváth and Z. Szabó, *Link Floer homology and the Thurston norm,* preprint (2006), math/0601618.
[24] P. Ozsváth and Z. Szabó, *A cube of resolutions for knot Floer homology,* preprint (2007), math/0705.3852.
[25] Y. Ni, *Knot Floer homology detects fibred knots,* preprint (2006), math.GT/0607156.
[26] J. Rasmussen, *Floer homology of surgeries on two-bridge knots,* Alg. Geom. Top. **2** (2002), 757-789.
[27] J. Rasmussen, *Floer homology and knot complements,* Ph. D. Thesis, Harvard University (2003), math/0306378.
[28] J. Rasmussen, "Knot polynomials and knot homologies," in *Geometry and topology of manifolds,* 261–280, Fields Inst. Commun. **47**, Amer. Math. Soc., Providence, 200
[29] A. Shumakovitch, *Torsion of the Khovanov homology*, preprint (2004), math/0405474.
[30] S. Sarkar and J. Wang, *A combinatorial description of some Heegaard Floer homologies,* preprint (2006), math/0607777.
[31] O. Viro, *Remarks on the definition of the Khovanov homology*, preprint (2002), math/0202199.

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Tight contact structures on the Weeks manifold

András I. Stipsicz

ABSTRACT. We construct tight contact structures on the Weeks manifold W, the closed hyperbolic 3–manifold with smallest volume. The contact structures are constructed through contact surgery diagrams involving contact (± 1)–surgeries. The tightness of these contact structures are verified by showing that the contact Ozsváth–Szabó invariants of the structures are nonzero.

1. Introduction

Suppose that Y is a closed oriented 3–manifold and ξ is a 2–plane field on Y which can be given as the kernel of a 1–form α satisfying $\alpha \wedge d\alpha > 0$. Such ξ is called a (positive, cooriented) *contact structure* on Y. A contact structure ξ is *tight* if Y does not contain any embedded disk D with the property that ξ is tangent to D along ∂D. Such disks are called *overtwisted disks*, and a contact structure containing such a disk is called overtwisted. According to the famous result of Eliashberg [5], overtwisted contact structures are determined by the homotopy type of the 2–plane field, hence do not capture geometric information of the underlying 3–manifold. Tight contact structures, however, are more geometric objects, and play prominent role in the study of low dimensional manifolds.

It is known that on a connected sum $Y_1 \# Y_2$ tight structures decompose uniquely as tight structures on the factors Y_1, Y_2. Therefore in understanding existence and classification questions we can restrict our attention to prime 3–manifolds. According to the solution of the Geometrization Conjecture, a prime 3–manifold Y either admits a hyperbolic metric, is Seifert fibered, or contains an essential torus (i.e., a 2–torus $T^2 \subset Y$ such that the embedding induces an injective homomorphism on the fundamental groups). According to [2, 8] a *toroidal* 3–manifold (i.e., a 3–manifold containing an essential torus) admits infinitely many different tight contact structures. (Moreover, by [2] an atoroidal manifold admits at most finitely many isotopy classes of tight contact structures.) The existence problem for Seifert fibered 3–manifolds was recently answered:

Theorem 1.1 ([14]). *A Seifert fibered 3–manifold admits a tight contact structure if and only if it is not orientation preserving diffeomorphic to the result of* $(2n-1)$*–surgery along the* $(2, 2n+1)$*–torus knot* $T_{2,2n+1} \subset S^3$ *for some* $n \in \mathbb{N}$. □

The proof of this theorem utilized the explicite surgery presentation of the Seifert fibered 3–manifolds. Using a surgery presentation, we were able to modify it to a contact surgery presentation of the same 3–manifold, where contact (± 1)–surgeries were performed along Legendrian knots in the standard contact 3–sphere. (For more on contact surgery see [3, 11].) Tigthness of the resulting contact structures were verified by the computation of the corresponding contact Ozsváth–Szabó invariants — their nonvanishing implied that the surgery diagrams gave rise to tight structures.

This approach cannot be applied for hyperbolic 3–manifolds in general. In this note we show that the Weeks manifold W, which is known to be the closed hyperbolic 3–manifold with smallest hyperbolic volume [6], admits tight contact structures. Recall that W is defined as surgery along the Whitehead link, with surgery coefficients $(5, \frac{5}{2})$. Following the convention of [1] we take the clasp at the Whitehead link in such a way that $(+1)$–surgery on one of its components turns the other one into the right–handed trefoil knot, while (-1)–surgery on the same component turns the other into the Figure–8 knot. For the surgery presentation of W see Figure 1, cf. also [4, page 249].

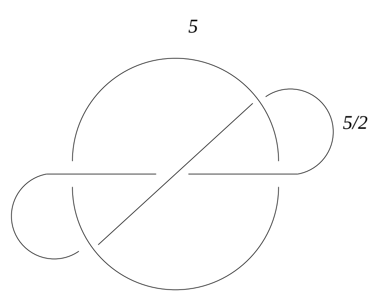

FIGURE 1. The Weeks manifold, presented as surgery along the Whitehead link

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2. Contact structures on the Weeks manifold W

In the following, through a sequence of surgery diagrams we describe contact structures on W with both of its orientations. Let us start with $-W$ (where the orientation of W is given by the diagram of Figure 1). A diagram for $-W$ can be given by considering the mirror image of the link of Figure 1, with the framings multiplied by (-1), as it is given by Figure 2. After an inverse slam dunk (cf. [7]) and an isotopy of the projection we can put these knots in Legendrian position in a way that a sequence of contact (-1)–surgeries is needed (when surgery coefficient is measured with respect to the contact framing), cf. Figure 2. Since on the

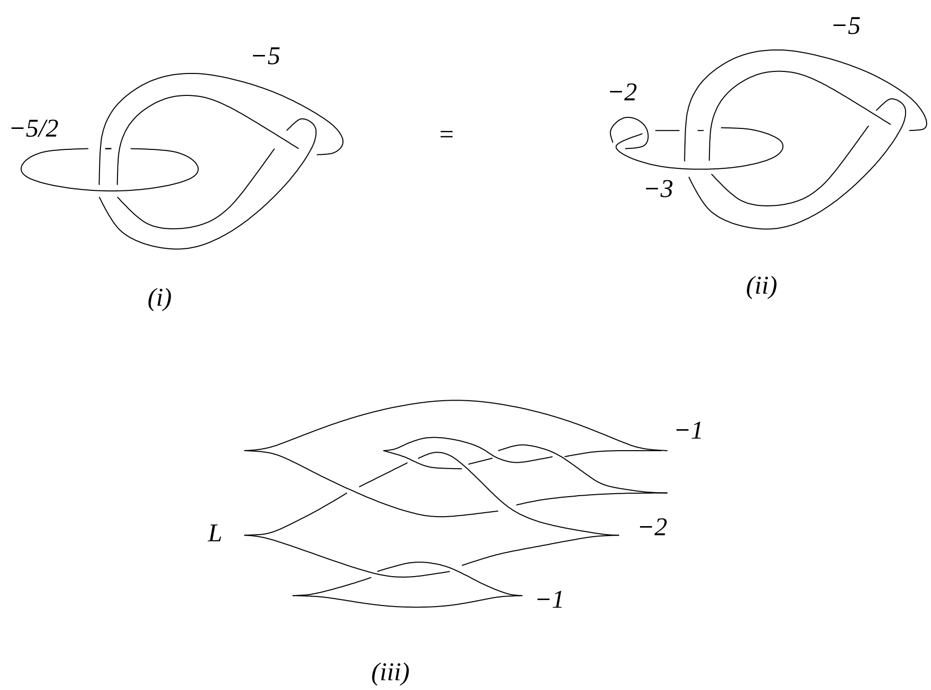

Figure 2. Stein fillable contact structures on $-W$

Legendrian knot L of Figure 2 (iii) we are free to choose the stabilization either on the right or on the left, we constructed two nonisotopic Stein fillable contact structures, which can be distinguished by the first Chern classes of their Stein fillings, cf. [9].

Now we turn our attention to the Weeks manifold W with the orientation given by Figure 1. A sequence of inverse slam dunks and blow–downs (cf. [7]) turns this diagram into the one given by Figure 3 (iii). Isotoping the link to Legendrian

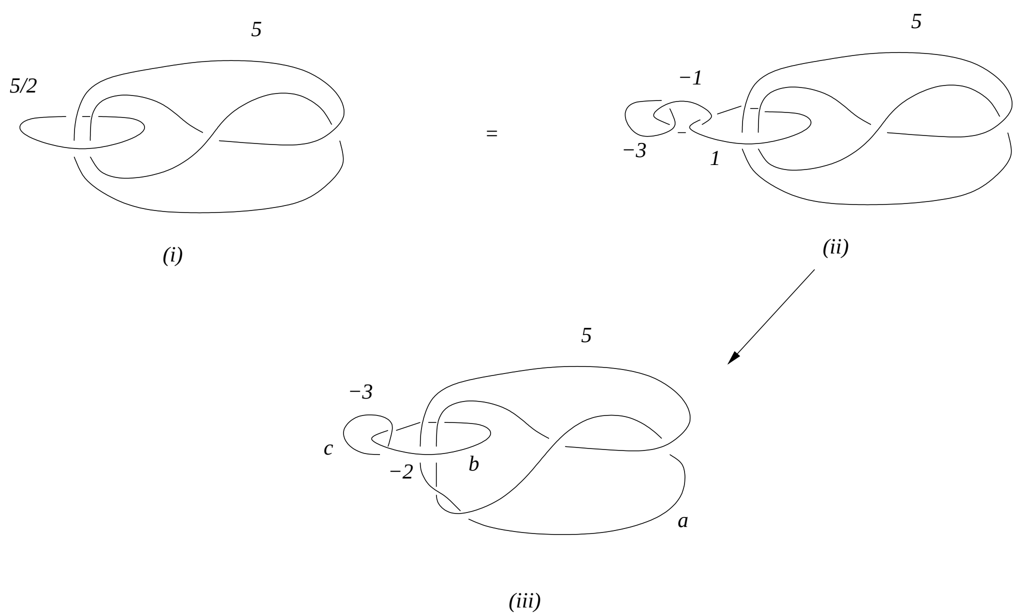

FIGURE 3. Another surgery diagrams for the Weeks manifold

position we get Figure 4. Now the standrad algorithm for turning a positive integer contact surgery into a sequence of contact (-1)–surgeries and a contact $(+1)$–surgery (as it is described in detail in [3]) yields contact surgery diagrams for W:

- stabilize the Legendrian unknot of Figure 4 with contact surgery coefficient (-2) once (either on the left or on the right side) and perform contact (-1)–surgery on the result, and

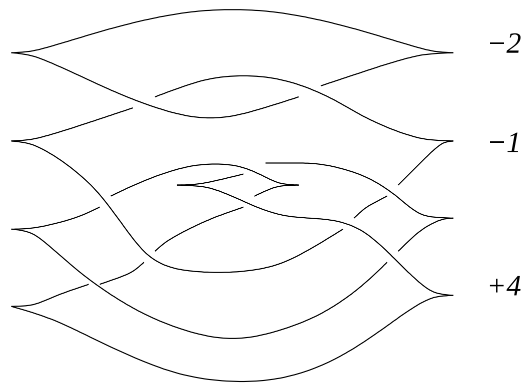

FIGURE 4. Contact surgery diagram for W

- consider the Legendian push-off of the $(+4)$–framed curve, stabilize it once (either on the right or on the left), and then push off the result two more times; perform contact $(+1)$–surgery on the orginial knot (which we will call L), while on the push–offs perform contact (-1)–surgeries.

Notice that in this algorithm we have two choices on whether we put the stabilizations to the left or to the right on the two curves, so in fact we produce four diagrams and hence four contact structures $\xi_{i,j}$ where $i,j \in \{l,r\}$ (corresponding to left or right stabilizations).

3. Contact Ozsváth–Szabó invariants

In [15, 16] Ozsváth and Szabó defined an invariant of a spin^c 3–manifold $(Y, \mathbf{t})$, which is now called the *Ozsváth–Szabó homology group* $\widehat{HF}(Y, \mathbf{t})$ of $(Y, \mathbf{t})$. The theory (which, for the sake of simplicity we consider here with $\mathbb{Z}_2$–coefficients) fits into the framework of a Topological Quantum Field Theory, in the sense that 4–dimensional spin^c cobordisms between spin^c 3–manifolds induce homomorphisms between the corresponding Ozsváth–Szabó homology groups. Moreover, these groups and maps provide exact triangles for surgery triples, as it is explained in [16], cf. also [10]. In addition, in [17] an invariant $c(Y, \xi)$ of a contact 3–manifold (Y, ξ) as an element of the group $\widehat{HF}(-Y, \mathbf{t}_\xi)$ is defined. (Here $\mathbf{t}_\xi$ denotes the spin^c structure induced by the contact structure ξ.) This invariant has the remarkable property that it vanishes for overtwisted contact strucutres (hence can be used to detect tightness), it is nonzero for Stein fillable contact structures and satisfies the following simple transformation rule: If (Y_L, ξ_L) is given as contact $(+1)$–surgery along the Legendrian knot $L \subset (Y, \xi)$, and the surgery induces the oriented cobordism X then

$$c(Y_L, \xi_L) = F_{-X}(c(Y, \xi)),$$

where F_{-X} is the map induced by the cobordism X, with its orientation reversed.

These invariants can be effectively computed in many cases when the contact structure is given through a contact surgery presentation, cf. [10, 11, 12]. Next we will determine the contact Ozsváth–Szabó invariants for the contact structures defined by the surgery diagrams we found on the Weeks manifold.

Theorem 3.1. *The contact structures $\xi_{i,j}$ $(i, j \in \{l, r\})$ defined by Figure 4 have nonvanishing contact Ozsváth–Szabó invariants, hence are all tight. In addition, the four structures are nonisotopic.*

Proof. Let us consider the single curve L of the contact surgery diagram defining ξ_{ij} on which contact $(+1)$–surgery is performed. Recall that by the recipe of [3], this curve results from the $(+4)$–framed Legendrian knot of Figure 4. It is not hard see that L is smoothly isotopic to the right–handed trefoil knot, and that $tb(L) = 1$. Therefore the main result of [10] (cf. also [13]) applies and shows that the contact Ozsváth–Szabó invariant of the resulting contact structure $(S^3_2(L), \xi_L)$ is nonzero. More explicitely, consider the cobordism X given by the handle attachment induced by the single surgery along L. The map F_{-X} fits in the exact triangle

$$\begin{tikzcd}[column sep=small] \widehat{HF}(S^3) \arrow[rr, "F_{-X}"] & & \widehat{HF}(S^3_{-2}(\overline{L})) \arrow[dl] \\ & \widehat{HF}(S^3_{-1}(\overline{L})) \arrow[ul] & \end{tikzcd}$$

where $\overline{L}$ is the mirror image of L (hence smoothly it is the left–handed trefoil knot). It is not hard to see that $\widehat{HF}(S^3_{-n}(\overline{L}) = \widehat{HF}(S^3_n(L)) = \mathbb{Z}_2^n$ for all $n > 0$, and it is known that $\widehat{HF}(S^3) = \mathbb{Z}_2$. Since the exactness of the triangle

$$\begin{tikzcd}[column sep=small] \mathbb{Z}_2 \arrow[rr, "F_{-X}"] & & \mathbb{Z}_2^2 \arrow[dl] \\ & \mathbb{Z}_2 \arrow[ul] & \end{tikzcd}$$

implies the injectivity of F_{-X}, the nonvanishing of the contact invariant $c(S^3, \xi_{st})$ of the standard contact 3–sphere and the identity $F_{-X}(c(S^3, \xi_{st})) = c(S^3_2(L), \xi_L)$ verifies the nonvanishing of $c(S^3_2(L), \xi_L)$.

Since the contact structures ξ_{ij} constructed on W are all contact (-1)–surgeries on ξ_L, the nonvanishing follows from [11, Corollary 3.6]. In fact, since a contact (-1)–surgery can be cancelled by a contact $(+1)$–surgery along the Legendrian push–off of the curve at hand [3, 18], we see that $(S^3_2(L), \xi_L)$ can be given as a sequence of contact $(+1)$–surgeries on $(W, \xi_{i,j})$. Therefore, for the corresponding cobordism U (induced by the surgeries viewed as handle attachments) we get

$$F_{-U}(c(W, \xi_{i,j})) = c(S^3_2(L), \xi_L),$$

and since the image is nonzero, we conclude that $c(W, \xi_{i,j}) \neq 0$, implying that these structures are all tight.

Finally, we show that the structures $\xi_{i,j}$ represent different spin^c structures, hence are nonisotopic. To this end, consider Figure 3(iii). Let the normal circle to the surgery curves a, b, c be denoted by μ_a, μ_b, μ_c. Then it is easy to see (cf. [7]) that $H_1(W;\mathbb{Z})$ can be presented as

$$\begin{aligned} H_1(W;\mathbb{Z}) &= \langle \mu_1, \mu_b, \mu_c \mid 5\mu_a = 0, -2\mu_b + \mu_c = 0, -3\mu_c + \mu_b = 0\rangle \\ &= \langle \mu_a, \mu_c \mid 5\mu_a = 5\mu_c = 0\rangle. \end{aligned}$$

From the surgery presentation of $\xi_{i,j}$ we get a 4–manifold X and a cohomology class $K \in H^2(X;\mathbb{Z})$, evaluating on the generators of the second homology corresponding to the surgery curves as their rotation numbers. As it is explained in [3], this characteristic cohomology class determines a spin^c structure $\mathbf{s}$ on X, with the property that the restriction of $\mathbf{s}$ to the boundary is exactly the spin^c structure of the contact structure given by the diagram. Suppose now that a choice of stabilizations is fixed, in other words, we picked one of $\xi_{i,j}$ $(i, j \in \{l, r\})$. After elementary Kirby calculus we can arrive to the diagram of Figure 3(iii), with the corresponding cohomology class K' evaluating on c as ± 1 (depending on the choice of the zig-zag), on b as 0, and on a as ± 3. Consequently the Poincaré dual of $c_1(\xi_{i,j})$, which is now an element of $H_1(W;\mathbb{Z})$ can be given in the presentation discussed above as $\pm 3a \pm c$. Since these elements are all different in $H_1(W;\mathbb{Z})$ we get that the contact structures $\xi_{i,j}$ induce different spin^c structures, hence are nonisotopic. $\square$

References

[1] D. Calegari, *Promoting essential laminations*, Invent. Math. **166** (2006) 583–643.

[2] V. Colin, E. Giroux and K. Honda, *On the coarse classification of tight contact structures*, Topology and geometry of manifolds (Athens, GA, 2001) 109–120. Proc. Sympos. Pure Math. **71**, 2003.

[3] F. Ding, H. Geiges and A. Stipsicz, *Surgery diagrams for contact 3–manifolds*, Turkish J. Math. **28** (2004) 41–74.

[4] N. Dunfield and W. Thurston, *The virtual Haken conjecture: Experiments and examples*, Geom. Topol. **7** (2003) 399–441.

[5] Y. Eliashberg, *Classification of overtwisted contact structures on 3–manifolds*, Invent. Math. **98** (1989) 623–637.

[6] D. Gabai, R. Meyerhoff and P. Milley, *Minimum volume cusped hyperbolic three-manifolds*, arXiv:0705.4325

[7] R. Gompf and A. Stipsicz, *4–manifolds and Kirby calculus*, Graduate Studies in Mathematics **20** AMS, 1999.

[8] K. Honda, W. Kazez and G. Matić, *On the Gabai–Eliashberg–Thurston theorem*, Comment. Math. Helv. **79** (2004) 502–515.

[9] P. Lisca and G. Matić, *Tight contact structures and Seiberg–Witten invariants*, Invent. Math. **129** (1997) 509–525.

[10] P. Lisca and A. Stipsicz, *Ozsváth–Szabó invariants and tight contact 3–manifolds, I*, Geom. Topol. **8** (2004) 925–945.

[11] P. Lisca and A. Stipsicz, *Ozsváth–Szabó invariants and tight contact 3–manifolds, II*, J. Differential Geom. **75** (2007) 109–141.

[12] P. Lisca and A. Stipsicz, *Ozsváth–Szabó invariants and tight contact 3–manifolds, III*, J. Symplectic Geometry, to appear, arXiv:math.SG/0505493.

[13] P. Lisca and A. Stipsicz, *Ozsváth–Szabó invariants and contact surgery*, Floer homology, gauge theory, and low-dimensional topology, 171–180, Clay Math. Proc. **5**, Amer. Math. Soc., Providence, RI, 2006.

[14] P. Lisca and A. Stipsicz, *On the existence of tight contact structures on Seifert fibered 3–manifolds*, arXiv:0709.0737

[15] P. Ozsváth and Z. Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. **159** (2004) 1027–1158.

[16] P. Ozsváth and Z. Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. **159** (2004) 1159–1245.

[17] P. Ozsváth and Z. Szabó, *Heegaard Floer homologies and contact structures*, Duke Math. J. **129** (2005) 39–61.

[18] A. Stipsicz, *Contact surgery and Heegaard Floer theory*, Floer homology, gauge theory, and low-dimensional topology, 143–170, Clay Math. Proc. **5**, Amer. Math. Soc., Providence, RI, 2006.

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Planar Contact Structures with Binding Number Three

Mehmet Fırat Arıkan

Abstract. In this article, we find the complete list of all contact structures (up to isotopy) on closed three-manifolds which are supported by an open book decomposition having planar pages with three (but not less) boundary components. We distinguish them by computing their first Chern classes and three dimensional invariants (whenever possible). Among these contact structures we also distinguish tight ones from those which are overtwisted.

1. Introduction

Let (M, ξ) be a closed oriented 3-manifold with the contact structure ξ, and let (S, h) be an open book (decomposition) of M which is compatible with ξ. In this case, we also say that (S, h) supports ξ (for the definitions of these terms see the next section). Based on Giroux's correspondence theorem (Theorem 2.3), two natural questions have been asked in [EO]:

(1) What is the possible minimal page genus $g(S)$ =genus(S)?

(2) What is the possible minimal number of boundary components of a page S with $g(S)$ minimal?

In [EO], two topological invariants $sg(\xi)$ and $bn(\xi)$ were defined to be the answers. More precisely, we have:

$$sg(\xi) = \min\{\ g(S) \mid (S, h) \text{ an open book decomposition supporting } \xi\},$$

called *the support genus* of ξ, and

$$bn(\xi) = \min\{\ |\partial S| \mid (S, h) \text{ an open book decomposition supporting } \xi \text{ and } g(S) = sg(\xi)\},$$

called *the binding number* of ξ. There are some partial results about these invariants. For instance, it is proved in [Et1] that if (M, ξ) is overtwisted, then $sg(\xi) = 0$.

Unlike the overtwisted case, there is not much known yet for $sg(\xi)$ if ξ is tight. The algorithm given in [Ar] finds a reasonable upper bound for $sg(\xi)$ using the given contact surgery diagram of ξ. However, there is no systematic way to obtain actual values of $sg(\xi)$ and $bn(\xi)$ yet.

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One of the ways to work on the above questions is to get a complete list of contact manifolds corresponding to a fixed support genus and a fixed binding number. To get such complete list, we consider all possible monodromy maps h. The first step in this direction is the following result given in [EO]. Throughout the paper $L(m,n)$ stands for the lens space obtained by $-m/n$ rational surgery on an unknot.

Theorem 1.1 ([EO])**.** *Suppose ξ is a contact structure on a 3-manifold M that is supported by a planar open book (i.e., $sg(\xi)=0$). Then*

(1) *If $bn(\xi)=1$, then ξ is the standard tight contact structure on S^3.*
(2) *If $bn(\xi)=2$ and ξ is tight, then ξ is the unique tight contact structure on the lens space $L(m,m-1)=L(m,-1)$ for some $m\in\mathbb{Z}_+\cup\{0\}$.*
(3) *If $bn(\xi)=2$ and ξ is overtwisted then ξ is the overtwisted contact structure on $L(m,1)$, for some $m\in\mathbb{Z}_+$, with $e(\xi)=0$ and $d_3(\xi)=-\frac{1}{4}m+\frac{3}{4}$ where $e(\xi)$ and $d_3(\xi)$ denotes the Euler class and d_3-invariant of ξ, respectively. When m is even then the refinement of $e(\xi)$ is given by $\Gamma(\xi)(\mathfrak{s})=\frac{m}{2}$ where $\mathfrak{s}$ is the unique spin structure on $L(m,1)$ that extends over a two handle attached to a μ with framing zero. Here we are thinking of $L(m,1)$ as $-m$ surgery on an unknot and μ is the meridian to the unknot.*

We remark that Theorem 1.1 gives the complete list of all contact 3-manifolds which can be supported by planar open books whose pages have at most 2 boundary components. Next step in this direction should be to find all contact 3-manifolds (M,ξ) such that $sg(\xi)=0$ and $bn(\xi)=3$. In the present paper, we will get all such contact structures, and also distinguish tight ones by looking at the monodromy maps of their corresponding open books (See Theorem 1.2 and Theorem 1.3). After the preliminary section (Section 2), we prove the main results in Section 3. Although some ideas in the present paper have been already given or mentioned in [EO], we will give their explicit statements and proofs in our settings. We finish this section by stating the main results.

Let Σ be the compact oriented surface with $|\partial\Sigma|=3$, and consider the boundary parallel curves a,b,c in Σ as in the Figure 1. Through out the paper, Σ will always stand for this surface whose abstract picture is given below. Let $Aut(\Sigma,\partial\Sigma)$ be the group of (isotopy classes of) diffeomorphisms of Σ which restrict to the identity on $\partial\Sigma$. (Such diffeomorphisms are automatically orientation-preserving).

It is known (see [Bi]) that

$$Aut(\Sigma,\partial\Sigma)=\mathbb{Z}\langle D_a\rangle\oplus\ \mathbb{Z}\langle D_b\rangle\oplus\ \mathbb{Z}\langle D_c\rangle\cong\mathbb{Z}^3$$

where D_a,D_b,D_c denote positive Dehn twists along the curves a,b,c given as in Figure 1. In the rest of the paper, we will not make any distinction between isotopy classes of arcs/curves/maps and the individual arcs/curves/maps.

We start with studying the group $Aut(\Sigma,\partial\Sigma)$ in details. Since generators commute with each other, we have that

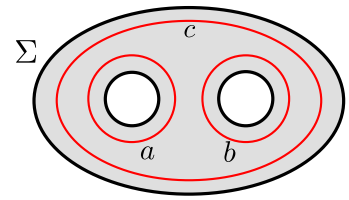


Figure 1. The surface Σ and the curves giving the generators of $Aut(\Sigma, \partial\Sigma)$.

$$Aut(\Sigma, \partial\Sigma) = \{{D_a}^p {D_b}^q {D_c}^r \,|\, p, q, r \in \mathbb{Z}\}.$$

For any given $p, q, r \in \mathbb{Z}$, let $Y(p, q, r)$ denote the smooth 3-manifold given by the smooth surgery diagram in Figure 2 (diagram on the left). It is an easy exercise to check that $Y(p, q, r)$ is indeed diffeomorphic to Seifert fibered manifold given in Figure 2 (diagram on the right).

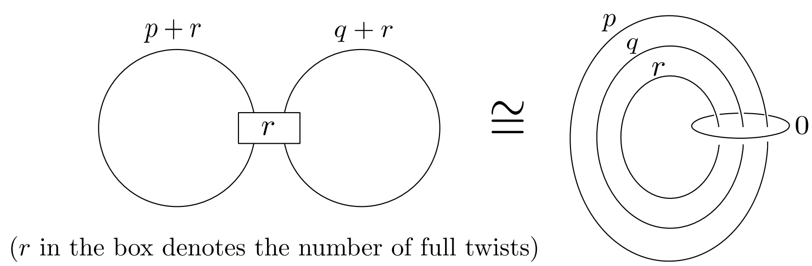


Figure 2. Seifert fibered manifold $Y(p, q, r)$.

Now we state the following theorem characterizing all closed contact 3-manifolds whose contact structures supported by open books $(\Sigma, \phi = {D_a}^p {D_b}^q {D_c}^r)$.

Theorem 1.2. *Let (M, ξ) be a contact manifold supported by the open book (Σ, ϕ) where $\phi = {D_a}^p {D_b}^q {D_c}^r \in Aut(\Sigma, \partial\Sigma)$ for fixed integers p, q, r. Then (M, ξ) is contactomorphic to $(Y(p, q, r), \xi_{p,q,r})$ where $\xi_{p,q,r}$ is the contact structure on $Y(p, q, r)$ given by the contact surgery diagram in Figure 3. Moreover,*

(1) *ξ is tight (in fact holomorphically fillable) if $p \geq 0, q \geq 0, r \geq 0$, and*
(2) *ξ is overtwisted otherwise.*

Remark 1.1. In Figure 3, if $r = 0$, then we completely delete the family corresponding to r from the diagram, so we are left with two families of Legendrian curves which do not link to each other, and so the contact surgery diagram gives a contact structure on the connected sum of two lens spaces. However, if $p = 0$ (or $q = 0$), then we replace

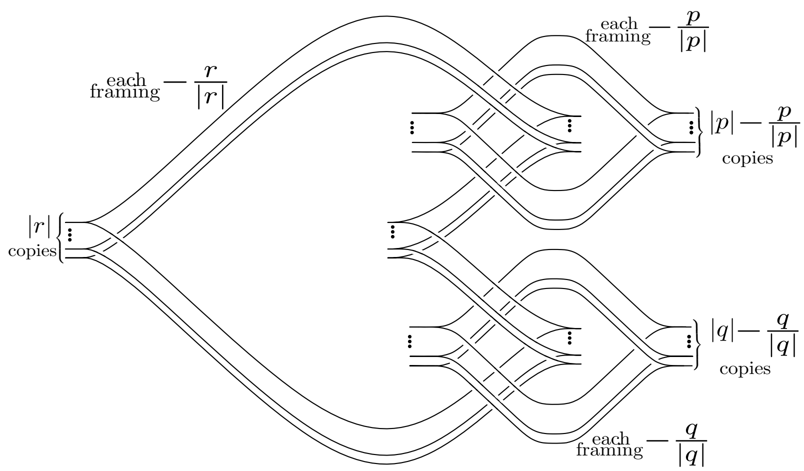


FIGURE 3. Contact manifold $(Y(p,q,r), \xi_{p,q,r})$.

the Legendrian family corresponding to p (or q) by a single Legendrian unknot with tb number equal to -1, and we do $(+1)$-contact surgery on the new unknot. Note also that Figure 3 is symmetric with respect to p and q. This reduces the number of cases in the proof of Theorem 1.3.

Of course not all $\xi_{p,q,r}$ have binding number three:

Theorem 1.3. *Let (M,ξ) be a closed contact 3-manifold for which $sg(\xi) = 0$ and $bn(\xi) = 3$. Then (M,ξ) is contactomorphic to some $(Y(p,q,r), \xi_{p,q,r})$ satisfying the following conditions:*

(1) *If $r = 0$, then $p \neq 1$ and $q \neq 1$.*
(2) *If $r = 1$, then $p \notin \{-1, 0\}$ and $q \notin \{-1, 0\}$.*
(3) *If $r = -1$, then $p \neq 1$ and $q \neq 1$.*
(4) *If $|r| \geq 2$, then $pq \neq -1$ and $(p,q) \notin \{(1,0),(0,1)\}$.*

Suppose that (M,ξ) is a closed contact 3-manifold with $sg(\xi) = 0$ and $bn(\xi) = 3$, and let $c_1 = c_1(\xi) \in H^2(M;\mathbb{Z})$ denote the first Chern class, and $d_3 = d_3(\xi)$ denote the 3-dimensional invariant (which lies in $\mathbb{Q}$ whenever c_1 is a torsion class in $H^2(M;\mathbb{Z})$). Using c_1 and d_3, we can distinguish these structures in most of the cases. In fact, we have either M is a lens space, or a connected sum of lens spaces, or a Seifert fibered manifold with three singular fibers. If one of the first two holds, then using the tables given in Section 3 and 4, one can get the complete list of all possible (M,ξ) without any repetition. That is, the contact structures in the list are all distinct pairwise and unique up to isotopy. On the other hand, if the third holds, we can also study them whenever c_1 is a torsion class. More discussion will be given in Section 4.

Acknowledgments. The author would like to thank Selman Akbulut, Çağrı Karakurt, Burak Özbağcı, and András Stipsicz for helpful conversations.

2. Preliminaries

2.1. Contact structures and open book decompositions

A 1-form $\alpha \in \Omega^1(M)$ on a 3-dimensional oriented manifold M is called a *contact form* if it satisfies $\alpha \wedge d\alpha \neq 0$. An *oriented contact structure* on M is then a hyperplane field ξ which can be globally written as kernel of a contact 1-form α. We will always assume that ξ is a *positive* contact structure, that is, $\alpha \wedge d\alpha > 0$. Two contact structures ξ_0, ξ_1 on a 3-manifold are said to be *isotopic* if there exists a 1-parameter family ξ_t ($0 \leq t \leq 1$) of contact structures joining them. We say that two contact 3-manifolds (M_1, ξ_1) and (M_2, ξ_2) are *contactomorphic* if there exists a diffeomorphism $f : M_1 \longrightarrow M_2$ such that $f_*(\xi_1) = \xi_2$. Note that isotopic contact structures give contactomorphic contact manifolds by Gray's Theorem. Any contact 3-manifold is locally contactomorphic to $(\mathbb{R}^3, \xi_0)$ where *standard contact structure* ξ_0 on $\mathbb{R}^3$ with coordinates (x, y, z) is given as the kernel of $\alpha_0 = dz + xdy$. The standard contact structure ξ_{st} on the 3-sphere $S^3 = \{(r_1, r_2, \theta_1, \theta_2) : r_1^2 + r_2^2 = 1\} \subset \mathbb{C}^2$ is given as the kernel of $\alpha_{st} = r_1^2 d\theta_1 + r_2^2 d\theta_2$. One basic fact is that $(\mathbb{R}^3, \xi_0)$ is contactomorphic to $(S^3 \setminus \{pt\}, \xi_{st})$. For more details on contact geometry, we refer the reader to [Ge], [Et3].

An *open book decomposition* of a closed 3-manifold M is a pair (L, f) where L is an oriented link in M, called the *binding*, and $f : M \setminus L \to S^1$ is a fibration such that $f^{-1}(t)$ is the interior of a compact oriented surface $S_t \subset M$ and $\partial \Sigma_t = L$ for all $t \in S^1$. The surface $S = S_t$, for any t, is called the *page* of the open book. The *monodromy* of an open book (L, f) is given by the return map of a flow transverse to the pages (all diffeomorphic to S) and meridional near the binding, which is an element $h \in Aut(S, \partial S)$, the group of (isotopy classes of) diffeomorphisms of S which restrict to the identity on ∂S . The group $Aut(S, \partial S)$ is also said to be the mapping class group of S, and denoted by $\Gamma(S)$.

An open book can also be described as follows. First consider the mapping torus

$$S(h) = [0, 1] \times S/(1, x) \sim (0, h(x))$$

where S is a compact oriented surface with $n = |\partial S|$ boundary components and h is an element of $Aut(S, \partial S)$ as above. Since h is the identity map on ∂S, the boundary $\partial S(h)$ of the mapping torus $S(h)$ can be canonically identified with n copies of $T^2 = S^1 \times S^1$, where the first S^1 factor is identified with $[0, 1]/(0 \sim 1)$ and the second one comes from a component of ∂S. Now we glue in n copies of $D^2 \times S^1$ to cap off $S(h)$ so that ∂D^2 is identified with $S^1 = [0, 1]/(0 \sim 1)$ and the S^1 factor in $D^2 \times S^1$ is identified with a boundary component of ∂S. Thus we get a closed 3-manifold

$$M = M_{(S,h)} := S(h) \cup_n D^2 \times S^1$$

equipped with an open book decomposition (S, h) whose binding is the union of the core circles in the $D^2 \times S^1$'s that we glue to $S(h)$ to obtain M. See [Gd], [Et2] for details.

2.2. Legendrian knots and contact surgery

A *Legendrian knot* K in a contact 3-manifold (M, ξ) is a knot that is everywhere tangent to ξ. Any Legendrian knot comes with a canonical *contact framing* (or *Thurston-Bennequin framing*), which is defined by a vector field along K that is transverse to ξ. We call (M, ξ) (or just ξ) *overtwisted* if it contains an embedded disc $D \approx D^2 \subset M$ with boundary $\partial D \approx S^1$ a Legendrian knot whose contact framing equals the framing it receives from the disc D. If no such disc exists, the contact structure ξ is called *tight*. Also if a contact 3-manifold (M, ξ) is the boundary of a Stein manifold (resp. a symplectic manifold) with certain compatibility conditions satisfied, then ξ is called *Stein (holomorphically) fillable* (resp. *symplectically fillable*). See [Et2] or [OS] for the complete definitions of fillability, and related facts. We will use the following fact later.

Theorem 2.1 ([EG])**.** *Any symplectically fillable contact structure is tight.*
($\Rightarrow$ *Any holomorphically fillable contact structure is tight.*)

For any $p, q \in \mathbb{Z}$, a contact (r)-surgery $(r = p/q)$ along a Legendrian knot K in a contact manifold (M, ξ) was first described in [DG]. It was proved in [Ho] that if $r = 1/k$ with $k \in \mathbb{Z}$, then the resulting contact structure is unique up to isotopy. In particular, a contact ± 1-surgery along a Legendrian knot K on a contact manifold (M, ξ) determines a unique surgered contact manifold which will be denoted by $(M, \xi)_{(K, \pm 1)}$.

The most general result along these lines is:

Theorem 2.2 ([DG])**.** *Every closed contact* 3*-manifold* (M, ξ) *can be obtained via contact* (± 1)*-surgery on a Legendrian link in* (S^3, ξ_{st})*.*

Any closed contact 3-manifold (M, ξ) can be described by a *contact surgery diagram* drawn in $(\mathbb{R}^3, \xi_0) \subset (S^3, \xi_{st})$. By Theorem 2.2, there is a contact surgery diagram for (M, ξ) such that the contact surgery coefficient of any Legendrian knot in the diagram is ± 1. For any oriented Legendrian knot K in $(\mathbb{R}^3, \xi_0)$, we compute the *Thurston-Bennequin number* $tb(K)$, and the *rotation number* $rot(K)$ as

$$tb(K) = bb(K) - (\text{\# of left cusps of K}),$$
$$rot(K) = \frac{1}{2}[(\text{\# of downward cusps}) - (\text{\# of upward cusps})]$$

where $bb(K)$ is the blackboard framing of K.

If a contact surgery diagram for (M, ξ) is given, we can also get the smooth surgery diagram for the underlying 3-manifold M. Indeed, for a Legendrian knot K in a contact surgery diagram, we have:

$$\text{Smooth surgery coefficient of } K = \text{Contact surgery coefficient of } K + tb(K)$$

For more details see [OS] and [Gm].

2.3. Compatibility and stabilization

A contact structure ξ on a 3-manifold M is said to be *supported by an open book* (L, f) if ξ is isotopic to a contact structure given by a 1-form α such that

(1) $d\alpha$ is a positive area form on each page $S \approx f^{-1}(\mathrm{pt})$ of the open book and
(2) $\alpha > 0$ on L (Recall that L and the pages are oriented.)

When this holds, we also say that the open book (L, f) is *compatible with the contact structure* ξ *on* M.

Definition 2.1. A positive (resp., negative) stabilization $S_K^+(S,h)$ (resp., $S_K^-(S,h)$) of an abstract open book (S,h) is the open book

(1) with page $S' = S \cup$ 1-handle and
(2) monodromy $h' = h \circ D_K$ (resp., $h' = h \circ D_K^{-1}$) where D_K is a right-handed Dehn twist along a curve K in S' that intersects the co-core of the 1-handle exactly once.

Based on the result of Thurston and Winkelnkemper [TW] which introduced open books into the contact geometry, Giroux proved the following theorem strengthening the link between open books and contact structures.

Theorem 2.3 ([Gi]). *Let M be a closed oriented 3-manifold. Then there is a one-to-one correspondence between oriented contact structures on M up to isotopy and open book decompositions of M up to positive stabilizations: Two contact structures supported by the same open book are isotopic, and two open books supporting the same contact structure have a common positive stabilization.*

Following fact was first implied in [LP], and then in [AO]. The given version below is due to Giroux and Matveyev. For a proof, see [OS].

Theorem 2.4. *A contact structure ξ on M is holomorphically fillable if and only if ξ is supported by some open book whose monodromy admits a factorization into positive Dehn twists only.*

For a given fixed open book (S,h) of a 3-manifold M, there exists a unique compatible contact structure up to isotopy on $M = M_{(S,h)}$ by Theorem 2.3. We will denote this contact structure by $\xi_{(S,h)}$. Therefore, an open book (S,h) determines a unique contact manifold $(M_{(S,h)}, \xi_{(S,h)})$ up to contactomorphism.

Taking a positive stabilization of (S,h) is actually taking a special Murasugi sum of (S,h) with the positive Hopf band (H^+, D_γ) where $\gamma \subset H^+$ is the core circle. Taking a Murasugi sum of two open books corresponds to taking the connect sum of 3-manifolds associated to the open books. The proofs of the following facts can be found in [Gd], [Et2].

Theorem 2.5. $(M_{S_K^+(S,h)}, \xi_{S_K^+(S,h)}) \cong (M_{(S,h)}, \xi_{(S,h)}) \# (S^3, \xi_{st}) \cong (M_{(S,h)}, \xi_{(S,h)})$.

Theorem 2.6. *Let (S,h) be an open book supporting the contact manifold (M,ξ). If K is a Legendrian knot on the page S of the open book, then*

$$(M,\xi)_{(K,\pm 1)} = (M_{(S,h\circ D_K^{\mp})}, \xi_{(S,h\circ D_K^{\mp})}).$$

2.4. Homotopy invariants of contact structures

The set of oriented 2−plane fields on a given 3-manifold M is identified with the space $Vect(M)$ of nonzero vector fields on M. $v_1, v_2 \in Vect(M)$ are called *homologous* (denoted by $v_1 \sim v_2$) if v_1 is homotopic to v_2 in $M \setminus B$ for some 3−ball B in M. The space $Spin^c(M)$ of all spin^c structures on M is the defined to be the quotient space $Vect(M)/\sim$. Therefore, any contact structure ξ on M defines a spin^c structure $\tau_\xi \in Spin^c(M)$ which depends only on the homotopy class of ξ. As the first invariant of ξ, we will use the first Chern class $c_1(\xi) \in H^2(M;\mathbb{Z})$ (considering ξ as a complex line bundle on M). For a spin^c structure τ_ξ, whose first Chern class $c_1(\tau_\xi)(:= c_1(\xi))$ is torsion, the obstruction to homotopy of two 2-plane fields (contact structures) both inducing τ_ξ can be captured by a single number. This obstruction is the 3-*dimensional invariant* $d_3(\xi)$ of ξ). To compute $d_3(\xi)$, suppose that a compact almost complex 4-manifold (X,J) is given such that $\partial X = M$, and ξ is the complex tangencies in TM, i.e., $\xi = TM \cap J(TM)$. Let $\sigma(X), \chi(X)$ denote the signature and Euler characteristic of X, respectively. Then we have

Theorem 2.7 ([Gm])**.** *If $c_1(\xi)$ is a torsion class, then the rational number*

$$d_3(\xi) = \frac{1}{4}\big(c_1^2(X,J) - 3\sigma(X) - 2\chi(X)\big)$$

is an invariant of the homotopy type of the 2-plane field ξ. Moreover, two 2-plane fields ξ_1 and ξ_2 with $\mathfrak{t}_{\xi_1} = \tau_{\xi_2}$ and $c_1(\tau_{\xi_i}) = c_1(\xi_i)$ a torsion class are homotopic if and only if $d_3(\xi_1) = d_3(\xi_2)$. □

As a result of this fact, if (M,ξ) is given by a contact ±1-surgery on a link, then we have

Corollary 2.8 ([DGS])**.** *Suppose that (M,ξ), with $c_1(\xi)$ torsion, is given by a contact (± 1)-surgery on a Legendrian link $\mathbb{L} \subset (S^3,\xi_{st})$ with $tb(K) \neq 0$ for each $K \subset \mathbb{L}$ on which we perform contact $(+1)$-surgery. Let X be a 4-manifold such that $\partial X = M$. Then*

$$d_3(\xi) = \frac{1}{4}\big(c^2 - 3\sigma(X) - 2\chi(X)\big) + s,$$

where s denotes the number of components in $\mathbb{L}$ on which we perform $(+1)$-surgery, and $c \in H^2(X;\mathbb{Z})$ is the cohomology class determined by $c(\Sigma_K) = rot(K)$ for each $K \subset \mathbb{L}$, and $[\Sigma_K]$ is the homology class in $H_2(X)$ obtained by gluing the Seifert surface of K with the core disc of the 2-handle corresponding K.

We use the above formula as follows: Suppose $\mathbb{L}$ has k components. Write $\mathbb{L} = \sqcup_i{}^k K_i$. By converting all contact surgery coefficients to the topological ones, and smoothing each cusp in the diagram, we get a framed link (call it $\mathbb{L}$ again) describing a simply connected 4-manifold X such that $\partial X = M$. Using this description, we compute

$$\chi(X) = 1 + k, \text{ and } \sigma(X) = \sigma(\mathcal{A}_{\mathbb{L}})$$

where $\mathcal{A}_{\mathbb{L}}$ is the linking matrix of $\mathbb{L}$. Using the duality, the number c^2 is computed as

$$c^2 = (PD(c))^2 = [b_1\ b_2 \cdots b_k]\mathcal{A}_{\mathbb{L}}[b_1\ b_2 \cdots b_k]^T$$

where $PD(c) \in H_2(X, \partial X; \mathbb{Z})$ is the Poincaré dual of c, the row matrix $[b_1\ b_2 \cdots b_k]$ is the unique solution to the linear system

$$\mathcal{A}_{\mathbb{L}}[b_1\ b_2 \cdots b_k]^T = [rot(K_1)\ rot(K_2) \cdots rot(K_k)]^T.$$

Here the superscript "T" denotes the transpose operation in the space of matrices. See [DGS], [Gm] for more details.

2.5. Right-veering diffeomorphisms

For a given compact oriented surface S with nonempty boundary ∂S, let $Dehn^+(S, \partial S) \subset Aut(S, \partial S)$ be the submonoid of product of all positive Dehn twists. In [HKM], another submonoid $Veer(S, \partial S)$ of all right-veering elements in $Aut(S, \partial S)$ was introduced and studied. They defined right-veering elements of $Aut(S, \partial S)$ as follows: Let α and β be isotopy classes (relative to the endpoints) of properly embedded oriented arcs $[0, 1] \to S$ with a common initial point $\alpha(0) = \beta(0) = x \in \partial S$. Let $\pi : \tilde{S} \to S$ be the universal cover of S (the interior of $\tilde{S}$ will always be $\mathbb{R}^2$ since S has at least one boundary component), and let $\tilde{x} \in \partial\tilde{S}$ be a lift of $x \in \partial S$. Take lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β with $\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{x}$. $\tilde{\alpha}$ divides $\tilde{S}$ into two regions – the region "to the left" (where the boundary orientation induced from the region coincides with the orientation on $\tilde{\alpha}$) and the region "to the right". We say that β is *to the right* of α if either $\alpha = \beta$ (and hence $\tilde{\alpha}(1) = \tilde{\beta}(1)$), or $\tilde{\beta}(1)$ is in the region to the right (see Figure 4).

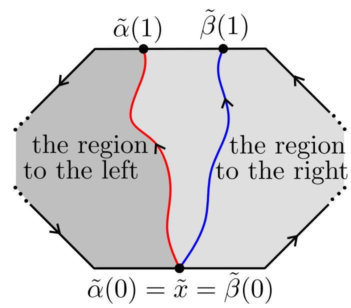


FIGURE 4. Lifts of α and β in the universal cover $\tilde{S}$.

Alternatively, isotop α and β, while fixing their endpoints, so that they intersect transversely (this include the endpoints) and with the fewest possible number of intersections. Assume that $\alpha \neq \beta$. Then in the universal cover $\tilde{S}$, $\tilde{\alpha}$ and $\tilde{\beta}$ will meet only at $\tilde{x}$. If not, subarcs of $\tilde{\alpha}$ and $\tilde{\beta}$ would cobound a disk D in $\tilde{S}$, and we could use an innermost disk argument on $\pi(D) \subset S$ to reduce the number of intersections of α and β by isotopy. Then β is to the right of α if $int(\tilde{\beta})$ lies in the region to the right. As an alternative to passing to the universal cover, we simply check to see if the tangent vectors $(\dot{\beta}(0), \dot{\alpha}(0))$ define the orientation on S at x.

Definition 2.2. Let $h : S \to S$ be a diffeomorphism that restricts to the identity map on ∂S. Let α be a properly embedded oriented arc starting at a basepoint $x \in \partial S$. Then h is *right-veering* (that is, $h \in Veer(S, \partial S)$) if for every choice of basepoint $x \in \partial S$ and every choice of α based at x, $h(\alpha)$ is to the right of α (at x). If C is a boundary component of S, we say is h is *right-veering with respect to* C if $h(\alpha)$ is to the right of α for all α starting at a point on C.

It turns out that $Veer(S, \partial S)$ is a submonoid and we have the inclusions:

$$Dehn^+(S, \partial S) \subset Veer(S, \partial S) \subset Aut(S, \partial S).$$

In [HKM], they proved the following theorem which is hard to use but still can be used to distinguish tight structure in some cases.

Theorem 2.9 ([HKM]). *A contact structure (M, ξ) is tight if and only if all of its compatible open book decompositions (S, h) have right-veering $h \in Veer(S, \partial S) \subset Aut(S, \partial S)$.*

3. The proofs of results

We first prove that the submonoids $Dehn^+(\Sigma, \partial\Sigma)$ and $Veer(\Sigma, \partial\Sigma)$ are actually the same in our particular case.

Lemma 3.1. *$Dehn^+(\Sigma, \partial\Sigma) = Veer(\Sigma, \partial\Sigma)$ for the surface Σ given in Figure 1.*

Proof. The inclusion $Dehn^+(S, \partial S) \subset Veer(S, \partial S)$ is true for a general compact oriented surface S with boundary (see Lemma 2.5. in [HKM] for the proof). Now, suppose that $\phi \in Veer(\Sigma, \partial\Sigma) \subset Aut(\Sigma, \partial\Sigma)$. Then we can write ϕ in the form

$$\phi = {D_a}^p {D_b}^q {D_c}^r \text{ for some } p, q, r \in \mathbb{Z}.$$

We will show that $p \geq 0, q \geq 0, r \geq 0$. Consider the properly embedded arc $\alpha \subset \Sigma$ one of whose end points is $x \in \partial\Sigma$ as shown in the Figure 5. Note that, for any $p, q, r \in \mathbb{Z}$, ${D_c}^r$ fixes α, and also any image ${D_a}^p {D_b}^q(\alpha)$ of α because c does not intersect any of these arcs. Assume at least one of p, q, or r is strictly negative. First assume that $p < 0$. Then consider two possible different images $\phi(\alpha) = {D_a}^p {D_b}^q(\alpha)$ of α corresponding to whether $q < 0$ or $q > 0$ (See Figure 5). Since we are not allowed to rotate any boundary component, clearly $\phi(\alpha)$ is to the left of α at the boundary point x. Equivalently, $\phi(\alpha)$ is not to the right of α at x which implies that h is not right-veering with respect to the boundary component parallel to a. Therefore, $\phi \notin Veer(\Sigma, \partial\Sigma)$ which is a contradiction. Now by symmetry, we are also done for the case $q < 0$. Finally, exactly the same argument (with a different choice of arc one of whose end points is on the boundary component parallel to the curve c) will work for the case when $r < 0$. $\square$

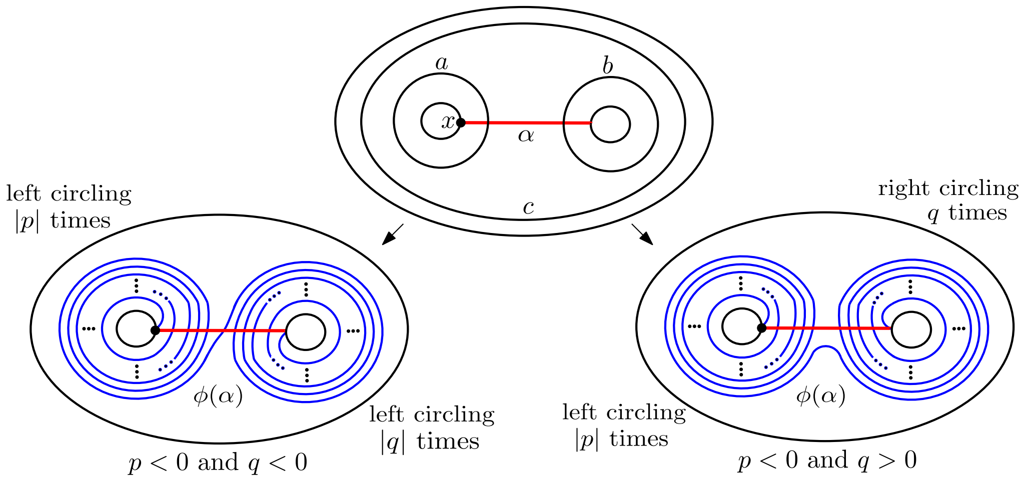


FIGURE 5. The arc α and its image $\phi(\alpha) = {D_a}^p {D_b}^q(\alpha)$.

Lemma 3.2. *Let (M, ξ) be a contact manifold. Assume that ξ is supported by (Σ, ϕ) where $\phi \in Aut(\Sigma, \partial\Sigma)$. Then ξ is tight if and only if ξ is holomorpfically fillable.*

Proof. Assume that ξ is tight. Since $\phi \in Aut(\Sigma, \partial\Sigma)$, there exists integers p, q, r such that $\phi = {D_a}^p {D_b}^q {D_c}^r$. As ξ is tight, the monodromy of any open book supporting ξ is right-veering by Theorem 2.9. In particular, we have $\phi \in Veer(\Sigma, \partial\Sigma)$ since (Σ, ϕ) supports ξ. Therefore, $\phi \in Dehn^+(\Sigma, \partial\Sigma)$ by Lemma 3.1, and so $p \geq 0, q \geq 0, r \geq 0$. Thus, ξ is holomorphically fillable by Theorem 2.4. Converse statement is a consequence of Theorem 2.1. □

Now, the following corollary of Lemma 3.2 is immediate:

Corollary 3.3. *Let (M, ξ) be a contact manifold. Assume that ξ is supported by (Σ, ϕ) where $\phi \in Aut(\Sigma, \partial\Sigma)$. Then*

$$\xi \textit{ is tight} \Longleftrightarrow \phi = {D_a}^p {D_b}^q {D_c}^r \textit{ with } p \geq 0, q \geq 0, r \geq 0. \qquad \square$$

Proof of Theorem 1.2. Let (M, ξ) be a contact manifold supported by the open book $(\Sigma, \phi_{p,q,r})$ where $\phi_{p,q,r} = {D_a}^p {D_b}^q {D_c}^r \in Aut(\Sigma, \partial\Sigma)$ for $p, q, r \in \mathbb{Z}$. As explained in [EO], $(M, \xi) = (M_{(\Sigma, \phi_{p,q,r})}, \xi_{(\Sigma, \phi_{p,q,r})})$ is given by the contact surgery diagram in Figure 6. Then we apply the algorithm given in [DG] and [DGS] to convert each rational coefficient into ± 1's, and obtain the diagram given in Figure 3.

To determine the topological (or smooth) type of (M, ξ), we start with the diagram in Figure 3. Then by converting the contact surgery coefficients into the smooth surgery

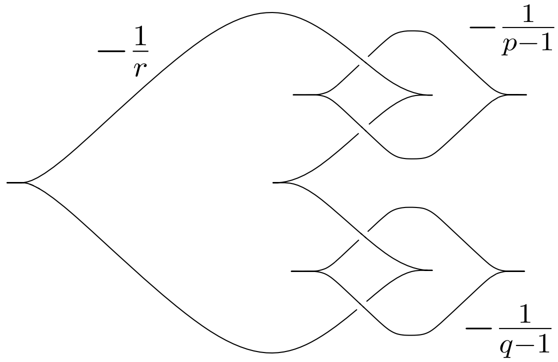


FIGURE 6. Contact surgery diagram corresponding to $(\Sigma, \phi_{p,q,r})$.

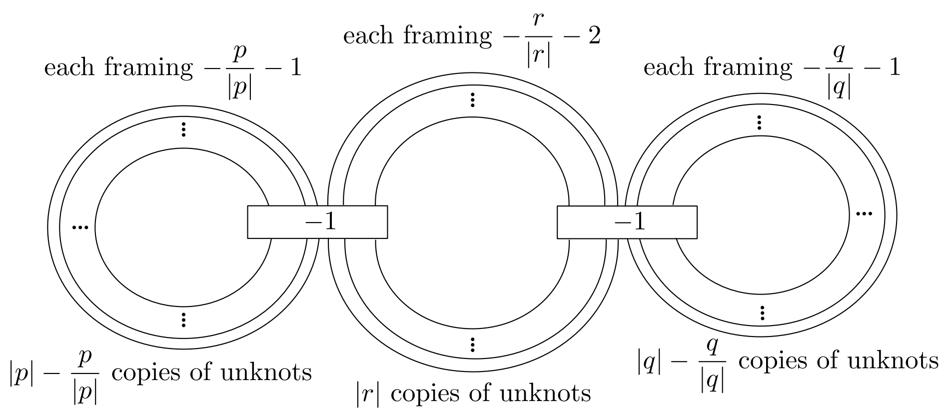


FIGURE 7. Smooth surgery diagram corresponding to Figure 3.

coefficients, we get the corresponding smooth surgery diagram in Figure 7 where each curve is an unknot.

Now we modify this diagram using a sequence of blow-ups and blow-downs. These operations do not change smooth type of M. We first blow up the diagram twice so that we unlink two -1 twists. Then we blow down each unknot in the most left and the most right families. Finally we blow down each unknot of the family in the middle. We illustrate these operations in Figure 8. To keep track the surgery framings, we note that each blow-up increases the framing of any unknot by 1 if the unknot passes through the corresponding twist box in Figure 7. So we get the first diagram in Figure 8. Blowing

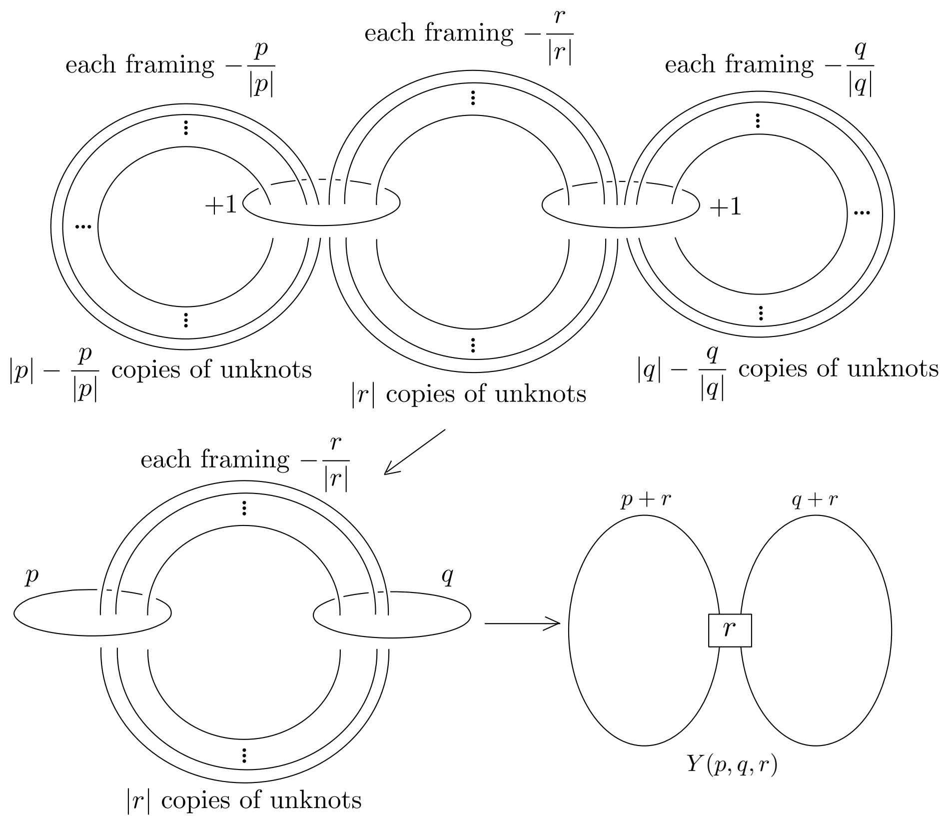


FIGURE 8. Squence of blow-ups and blow-downs.

each member down on the left (resp. right) decreases the framing of the left (resp. right) +1-unknot by $-\frac{p}{|p|}$ (resp. $-\frac{q}{|q|}$). Since there are $|p| - \frac{p}{|p|}$ blow-downs on the left and $|q| - \frac{q}{|q|}$ blow-downs on the right, we get the second diagram in Figure 8. Finally, if we blow down each $(-\frac{r}{|r|})$-unknot in the middle family, we get the last diagram. Note that each blow-down decreases the framing by $-\frac{r}{|r|}$, and introduces a $\frac{r}{|r|}$ full twist. Hence, we showed that (M, ξ) is contactomorphic to $(Y(p,q,r), \xi_{p,q,r})$. The statements (1) and (2) are the consequences of Corollary 3.3. $\square$

We now examine the special case where $Y(p,q,r)$ is homeomorphic to 3-sphere S^3. The following lemma lists all planar contact structures on S^3 with binding number less than or equal to three.

Lemma 3.4. *Suppose that $(Y(p,q,r),\xi_{p,q,r})$ is contactomorphic to (S^3,ξ) for some contact structure ξ on S^3. Then Table 1 lists all possible values of (p,q,r), the corresponding ξ (in terms of the d_3-invariant), and its binding number.*

r	p	q	$d_3(\xi)$	$bn(\xi)$
-3	-2	1	$-1/2$	3
-2	-3	1	$-1/2$	3
-1	1	any q	$1/2$	2
-1	3	2	$3/2$	3
0	1	1	$-1/2$ (tight)	1
0	-1	-1	$3/2$	3
0	1	-1	$1/2$	2
1	0	1	$-1/2$ (tight)	1
1	-1	any q	$1/2$	2
1	-2	-3	$-1/2$	3
2	3	-1	$3/2$	3
3	2	-1	$3/2$	3
$\lvert r\rvert \geq 2$	1	-1	$1/2$	2

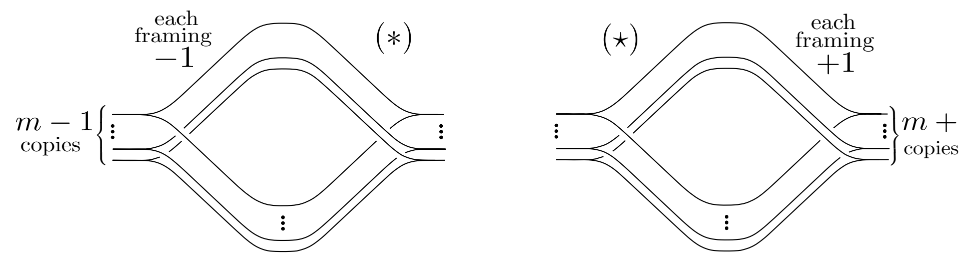
TABLE 1. All planar contact structures on S^3 with binding number ≤ 3.

Proof. The proof is the direct consequence of the discussion given in the proof of Lemma 5.5 in [EO]. We remark that the interchanging p and q does not affect the contact structure in Figure 3, so we do not list the possibilities for (p,q,r) that differ by switching p and q. Note that in Table 1 there are only two contact structures (up to isotopy) on S^3 with binding number 3, namely, the ones with d_3-invariants $-1/2$ and $3/2$. □

Proof of Theorem 1.3. We will use the results of Theorem 1.1, Theorem 1.2, and Lemma 3.4. Consider the 3-sphere S^3 in Theorem 1.1 as the lens space $L(1,\pm 1)$. By Theorem 1.1, for any contact manifold (Y,η) with $sg(\eta)=0$ and $bn(\eta)\leq 2$, we have either

(1) $(Y,\eta)\cong(S^3,\xi_{st})$ if $bn(\eta)=1$,
(2) $(Y,\eta)\cong(L(m,-1),\eta_m)$ for some $m\geq 2$ if $bn(\eta)=2$, and η is tight,
(3) $(Y,\eta)\cong(L(m,1),\eta_m)$ for some $m\geq 0$ if $bn(\eta)=2$, and η is overtwisted (for $m\neq 0$).

where η_m is the contact structure on the lens space $L(m,-1)$ (or $L(m,1)$) given by the contact surgery diagram consisting of a single family of Legendrian unknots (with Thurston-Benequen number -1) such that each member links all the other members of the family once, and each contact surgery coefficient is -1 (if η_m is tight) or $+1$ (if η_m is overtwisted). These are illustrated by the diagrams $(*)$ and $(\star)$ in Figure 9, respectively. Notice the exceptional cases: $m=1$ in $(*)$, and $m=0$ in $(\star)$.



$(Y,\eta) \cong (L(m,-1),\eta_m)$
η_m always tight, $m \geq 1$
$m=1 \Rightarrow$ empty diagram

$(Y,\eta) \cong (L(m,1),\eta_m)$
η_m overtwisted if $m \geq 1$
η_m tight if $m=0$

Figure 9. Contact surgery diagrams for (Y,η).

Now, if (M,ξ) is a contact manifold with $sg(\xi)=0$ and $bn(\xi)=3$, then by the definitions of these invariants there exists an open book (Σ,ϕ) supporting ξ. Therefore, by Theorem 1.2, (M,ξ) is contactomorphic to $(Y(p,q,r),\xi_{p,q,r})$ for some $p,q,r \in \mathbb{Z}$, and the contact surgery diagram of ξ is given in Figure 3. However, p,q,r can not be arbitrary integers because there are several cases where the diagram in Figure 3 reduces to either $(*)$ or $(\star)$ in Figure 9 for some m. So for those values of p,q,r, (M,ξ) can not be contactomorphic to $(Y(p,q,r),\xi_{p,q,r}) \cong (Y,\eta)$ because $bn(\xi)=3 \neq 2 \geq bn(\eta)$. Therefore, we have to determine those cases.

If $|p| \geq 2$ and $|q| \geq 2$, then the only triples (p,q,r) giving $L(m,\pm 1)'s$ are $(-2,q,1)$ and $(2,q,-1)$. Furthermore, if we assume also that $|r|>1$, then the Seifert fibered manifolds $Y(p,q,r)$ are not homeomorphic to even a lens space $L(m,n)$ for any m,n (for instance, see Chapter 5 in [Or]). As a result, we immediately obtain $bn(\xi_{p,q,r})=3$ for $|p| \geq 2$ and $|q| \geq 2$ and $|r| \geq 2$. Therefore, to finish the proof of the theorem, it is enough to analyze the cases where $|p|<2$ or $|q|<2$, and the cases $(-2,q,1)$ and $(2,q,-1)$ for any q. As we remarked before, we do not need to list the possibilities for (p,q,r) that differ by switching p and q. We first consider $r=0$, ± 1, ± 2, and then the cases $r>2$ and $r<-2$. In Table 2 - 8, we list all possible (M,ξ) for each of these cases.

Remark 3.1. To determine the binding number $bn(\xi)$ in any row of any table below, we simply first check the topological type of the manifold under consideration. If $M \approx S^3$, we determine the corresponding binding number using Table 1. If the topological type is not $L(m,1)$ or $L(m,-1)$, then we immediately get that $bn(\xi)=3$. If $M \approx L(m,1)$ with $m>1$, then we first compute $c_1(\xi)$. If $c_1(\xi) \neq 0$, then $bn(\xi)=3$ as $c_1(\eta_m)=0$ for any η_m given above. If $c_1(\xi)$=0, we compute the $d_3(\xi)$ using the 4-manifold defined by the surgery diagram in Figure 7. (Indeed, we can use the formula for d_3 given in Corollary 2.8 as long as $c_1(\xi)$ is torsion. In particular, whenever $H^2(M)$ is finite, then d_3 is computable). Then if $d_3(\xi)=d_3(\eta_m)=(-m+3)/4$, then ξ is isotopic to η_m which implies that $bn(\xi)=2$ by Theorem 1.1. Otherwise $bn(\xi)=3$. In the case that $M \approx L(m,-1)$ with $m>1$, we first ask if ξ is tight. If it is tight (which is the case if and only if $p \geq 0, q \geq 0, r \geq 0$), then $bn(\xi)=2$ (again by Theorem 1.1) since the tight structure on $L(m,-1)$ is unique (upto isotopy). If it is overtwisted (which is the case if and only if at least one of p,q,r is negative), then $bn(\xi)=3$ because ξ is not covered in Theorem 1.1. As a final remark, sometimes the contact structure ξ can be viewed as a positive stabilization of some η_m. For these cases we immediately obtain that $bn(\xi)=2$ because positive stabilizations do not change the isotopy classes of contact structures.

To compute the d_3-invariant of $\xi_{p,q,r}$ (for $c_1(\xi_{p,q,r})$ torsion), we will use the $(n+1)\times(n+1)$ matrices A_n $(n \geq 1)$, B_n $(n \geq 1)$, and C_n $(n \geq 4)$ given below. It is a standard exercise to check that

(1) $\sigma(A_n)=n-1$ if $n \geq 1$, and $\sigma(C_n)=n-1$ if $n \geq 4$.

(2) $\sigma(B_n)=n-3$ if $n \geq 3$, and $\sigma(B_n)=0$ if $n=1,2$.

(3) The system $A_n[\mathbf{b}]_{n+1}^T=[\mathbf{0}]_{n+1}^T$ has trivial solution $[\mathbf{b}]_{n+1}^T=[\mathbf{0}]_{n+1}^T$ where $[\mathbf{b}]_{n+1}=[b_1\ b_2 \cdots b_{n+1}]$, $[\mathbf{0}]_{n+1}=[0\ 0 \cdots 0]$ are $(n+1)\times 1$ row matrices.

$$A_n = \begin{pmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ -1 & -1 & \ddots & & \vdots \\ \vdots & \vdots & & 0 & -1 \\ -1 & -1 & \cdots & -1 & 0 \end{pmatrix} \quad B_n = \begin{pmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ -1 & -1 & \ddots & & \vdots \\ \vdots & \vdots & & 0 & -1 \\ -1 & -1 & \cdots & -1 & -2 \end{pmatrix} \quad C_n = \begin{pmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ -1 & -1 & \ddots & & \vdots \\ \vdots & \vdots & & 0 & -1 \\ -1 & -1 & \cdots & -1 & 2 \end{pmatrix}$$

In some cases, A_n appears (as a block matrix) in the linking matrix $\mathcal{L}_{p,q,r}$ of the framed link $\mathbb{L}_{p,q,r}$ given in Figure 7. On the other hand, B_n and C_n are very handy when we diagonalize $\mathcal{L}_{p,q,r}$ to find its signature. As we discussed before, the link $\mathbb{L}_{p,q,r}$ defines a 4-manifold $X_{p,q,r}$ with $\partial X=M$. So we have

$$\begin{aligned} \sigma(X_{p,q,r}) &= \sigma(\mathcal{L}_{p,q,r}), \\ \chi(X_{p,q,r}) &= 1+(\#\text{ of components of }\mathbb{L}_{p,q,r}), \\ c^2 &= [\mathbf{b}]_k \mathcal{L}_{p,q,r} [\mathbf{b}]_k^T \end{aligned}$$

where $[\mathbf{b}]_k^T$ is the solution to the linear system $\mathcal{L}_{p,q,r}[\mathbf{b}]_k^T = [rot(K_1)\ rot(K_2) \cdots rot(K_k)]^T$ with $K_1, K_2, \cdots K_k$ being the components of $\mathbb{L}_{p,q,r}$.

To compute the first Chern class $c_1(\xi_{p,q,r}) \in H^2(M)$, note that in Figure 3, the rotation number of any member in the family corresponding to r is ± 1 (depending on how we orient them). We will always orient them so that their rotation numbers are all $+1$. On the other hand, the rotation number is 0 for any member in the family corresponding to p and q. Therefore, $c_1(\xi_{p,q,r}) = PD^{-1}(\mu_1 + \mu_2 + \cdots + \mu_{|r|})$ where $\mu_i \in H_1(M)$ is the class of the meridian of the Legendrian knot K_i in the family corresponding to r. Then we compute $H_1(M)$ (which is isomorphic to $H^2(M)$ by Poincaré duality) as

$$H_1(M) = \langle\ \mu_1, \mu_2, \cdots, \mu_k |\ \mathcal{L}_{p,q,r}[\mu]_k^T = [\mathbf{0}]_k^T\ \rangle$$

where $[\mu]_k = [\mu_1\ \ \mu_2 \cdots \mu_k]$ is the $k \times 1$ row matrix. The final step is to understand $PD(c_1(\xi_{p,q,r})) = \mu_1 + \mu_2 + \cdots + \mu_{|r|}$ in this presentation of $H_1(M)$.

r	p	q	resulting M	$bn(\xi)$	diagram for ξ	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
0	-1	-1	S^3	3	Figure 3	$0 \in \{0\}$	$3/2$
0	-1	0	$S^3 \# S^1 \times S^2$	3	Figure 3	$0 \in \mathbb{Z}$	1
0	-1	1	S^3	2	$(\star)$ $m = 1$	$0 \in \{0\}$	$1/2$
0	-1	$q \geq 2$	$S^3 \# L(q, -1)$	3	Figure 3	$0 \in \mathbb{Z}_q$	$(q+1)/4$
0	-1	$q \leq -2$	$S^3 \# L(\|q\|, 1)$	3	Figure 3	$0 \in \mathbb{Z}_{\|q\|}$	$(-\|q\|+7)/4$
0	0	0	$\#_2 S^1 \times S^2$	3	Figure 3	$0 \in \mathbb{Z} \oplus \mathbb{Z}$	$1/2$
0	0	1	$S^1 \times S^2 \# S^3$	2	$(\star)$ $m = 0$	$0 \in \mathbb{Z}$	0
0	0	$q \geq 2$	$S^1 \times S^2 \# L(q, -1)$	3	Figure 3	$0 \in \mathbb{Z} \oplus \mathbb{Z}_q$	$(q-1)/4$
0	0	$q \leq -2$	$S^1 \times S^2 \# L(\|q\|, 1)$	3	Figure 3	$0 \in \mathbb{Z} \oplus \mathbb{Z}_{\|q\|}$	$(-\|q\|+5)/4$
0	1	1	S^3	1	$(*)$ $m = 1$	$0 \in \{0\}$	$-1/2$
0	1	$q \geq 2$	$S^3 \# L(q, -1)$	2	$(*)$ $m = q$	$0 \in \mathbb{Z}_q$	$(q-3)/4$
0	1	$q \leq -2$	$S^3 \# L(\|q\|, 1)$	2	$(\star)$ $m = \|q\|$	$0 \in \mathbb{Z}_{\|q\|}$	$(-\|q\|+3)/4$

Table 2. The case $r = 0$ ($|p| < 2$ or $|q| < 2$).

In **Table 2**, we need to compute the binding number $bn(\xi)$ for the rows 5, 12. For the other rows, see Remark 3.1.

• If $p = -1, q \leq -2, r = 0$, we need to compute $d_3(\xi_{-1,q,0})$ as $c_1(\xi_{-1,q,0}) = 0$: We have

$$\mathcal{L}_{-1,q,0} = \begin{pmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & A_{|q|+1} \end{pmatrix}.$$

The contact structure $\xi_{-1,q,0}$, and $\mathbb{L}_{-1,q,0}$ describing $X_{-1,q,0}$ are given in Figure 10. We compute that $s = |q|+3$, $c^2 = 0$, $\chi(X_{-1,q,0}) = |q|+4$, and $\sigma(X_{-1,q,0}) = \sigma(A_1)+\sigma(A_{|q|}) = 0+|q|-1 = |q|-1$, and so we obtain $d_3(\xi_{-1,q,0}) = (-|q|+7)/4$ by Corollary 2.8. Therefore, $\xi_{-1,q,0}$ is not isotopic to $\eta_{|q|}$ as $d_3(\eta_{|q|}) = (-|q|+3)/4$. Hence, $bn(\xi_{-1,q,0}) = 3$ for any $q \leq -2$ by Theorem 1.1.

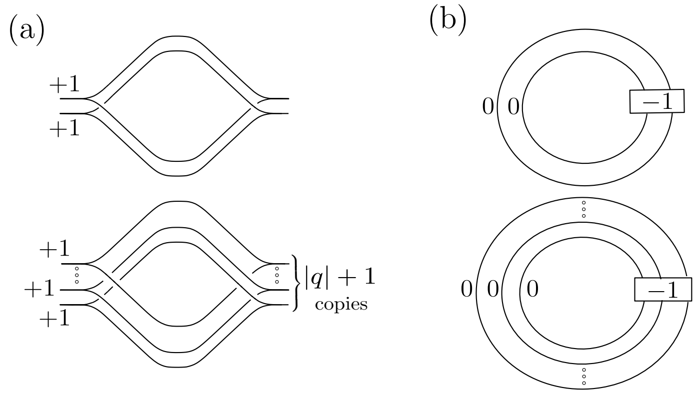


FIGURE 10. (a) The contact structure $\xi_{-1,q,0}$ on $S^3\#L(|q|,1) \approx L(|q|,1)$, (b) The corresponding framed link $\mathbb{L}_{-1,q,0}$.

• If $p = 1, q \leq -2, r = 0$, we have $(\Sigma, \phi_{1,q,0}) = S_a^+(H^+, D_b^q)$ (recall the identification of Σ and the curves a, b, c in Figure 1). Therefore, $\xi_{1,q,0} \cong \eta_{|q|}$ since (H^+, D_b^q) supports the overtwisted structure $\eta_{|q|}$ on $L(|q|,1)$. Hence, $bn(\xi_{1,q,0}) = 2$ for $q \leq -2$.

In **Table 3**, we need to compute the binding number $bn(\xi)$ for the rows 1 and 9. For the other rows, see Remark 3.1.

• If $p = -2, q \leq -4, r = 1$, let K_i's be the components (with the given orientations) of $\mathbb{L}_{-2,q,1}$ as in Figure 11. Then we obtain the linking matrix

$$\mathcal{L}_{-2,q,1} = \begin{pmatrix} -3 & -1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & -1 & 0 & \cdots & 0 \\ -1 & -1 & 0 & -1 & 0 & \cdots & 0 \\ -1 & -1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 0 & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & A_{|q|} & \\ \cdot & \cdot & \cdot & \cdot & & & \\ -1 & 0 & 0 & 0 & & & \end{pmatrix}$$

r	p	q	resulting M	$bn(\xi)$	diagram for ξ	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
1	-2	$q \leq -4$	$L(\lvert q+2\rvert, 1)$	3	Figure 3	$\lvert q\rvert - 4 \in \mathbb{Z}_{\lvert q\rvert-2}$	$\frac{-q^2-7q-14}{-4q-8}$
1	-2	-3	S^3	3	Figure 3	$0 \in \{0\}$	$-1/2$
1	-2	-2	$S^1 \times S^2$	3	Figure 3	$-2 \in \mathbb{Z}$	$\notin Q$
1	-2	$q \geq 2$	$L(q+2, -1)$	3	Figure 3	$q \in \mathbb{Z}_{q+2}$	$\frac{q^2+q+2}{4q+8}$
1	-1	any q	S^3	2	$(\star)$ $m=1$	$0 \in \{0\}$	$1/2$
1	0	0	$S^1 \times S^2$	2	$(\star)$ $m=0$	$0 \in \mathbb{Z}$	0
1	0	1	S^3	1	$(*)$ $m=1$	$0 \in \{0\}$	$-1/2$
1	0	$q \geq 2$	$L(q, -1)$	2	$(*)$ $m=q$	$0 \in \mathbb{Z}_q$	$(q-3)/4$
1	0	$q \leq -1$	$L(\lvert q\rvert, 1)$	2	$(\star)$ $m=\lvert q\rvert$	$0 \in \mathbb{Z}_{\lvert q\rvert}$	$(-\lvert q\rvert+3)/4$
1	1	-2	$L(3, -1)$	3	Figure 3	$1 \in \mathbb{Z}_3$	$1/3$
1	1	1	$L(3, 1)$	3	Figure 3	$1 \in \mathbb{Z}_3$	$-1/3$
1	1	$q \leq -3$	$L(2q+1, -q-1)$	3	Figure 3	$\lvert q\rvert - 1 \in \mathbb{Z}_{2\lvert q\rvert-1}$	$\frac{-q^2-4q-2}{-4q-2}$
1	1	$q \geq 2$	$L(2q+1, -q-1)$	3	Figure 3	$q+1 \in \mathbb{Z}_{2q+1}$	$\frac{q^2-2q-1}{4q+2}$

TABLE 3. The case $r = 1$, $|p| < 2$ or $|q| < 2$ (and the case $(p, q, r) = (-2, q, 1)$).

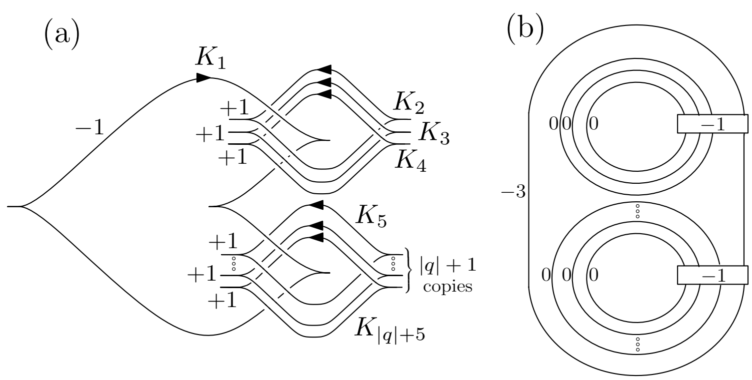


FIGURE 11. (a) The overtwisted contact structure $\xi_{-2,q,1}$ on $L(|q+2|, 1)$, (b) The corresponding framed link $\mathbb{L}_{-2,q,1}$.

It is not hard to see that

$H_1(M) = \langle\ \mu_1, \mu_2, \cdots, \mu_{|q|+5} |\ \mathcal{L}_{-2,q,1}[\mu]^T_{|q|+5} = [\mathbf{0}]^T_{|q|+5}\ \rangle = \langle\ \mu_2 |\ (|q|-2)\mu_2 = 0\ \rangle \cong \mathbb{Z}_{|q|-2}$,

and $\mu_1 = (|q| - 4)\mu_2$. Therefore,

$$c_1(\xi_{-2,q,1}) = PD^{-1}(\mu_1) = PD^{-1}(|q| - 4)\mu_2 = |q| - 4 \in \mathbb{Z}_{|q|-2}.$$

Thus, if $q < -4$, then $\xi_{-2,q,1}$ is not isotopic to $\eta_{|q+2|}$ as $c_1(\eta_{|q+2|}) = 0$ implying that $bn(\xi_{-2,q,1}) = 3$ by Theorem 1.1. If $q = -4$, we compute that $d_3(\xi_{-2,-4,1}) = -1/4 \neq 1/4 = d_3(\eta_2)$, so $bn(\xi_{-2,-4,1}) = 3$.

• If $p = 0, q \leq -1, r = 1$, we have $(\Sigma, \phi_{0,q,1}) = S^+_c(H^+, D^q_b)$ (again recall the identification of Σ and the curves a, b, c in Figure 1). Therefore, $\xi_{0,q,1} \cong \eta_{|q|}$ since (H^+, D^q_b) supports the overtwisted structure $\eta_{|q|}$ on $L(|q|, 1)$. Hence, $bn(\xi_{0,q,1}) = 2$ for $q < 0$.

r	p	q	resulting M	$bn(\xi)$	diagram for ξ	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
-1	2	$q \geq 4$	$L(q-2, -1)$	3	Figure 3	$q - 4 \in \mathbb{Z}_{q-2}$	$\frac{-q^2+3q-6}{-4q+8}$
-1	2	3	S^3	3	Figure 3	$0 \in \{0\}$	$3/2$
-1	2	2	$S^1 \times S^2$	3	Figure 3	$-2 \in \mathbb{Z}$	$\notin Q$
-1	2	$q \leq -2$	$L(\|q-2\|, 1)$	3	Figure 3	$\|q\| \in \mathbb{Z}_{\|q\|+2}$	$\frac{-q^2-3q+6}{-4q+8}$
-1	1	any q	S^3	2	$(\star)$ $m = 1$	$0 \in \{0\}$	$1/2$
-1	0	0	$S^1 \times S^2$	3	Figure 3	$0 \in \mathbb{Z}$	1
-1	0	-1	S^3	3	Figure 3	$0 \in \{0\}$	$3/2$
-1	0	$q > 1$	$L(q, -1)$	3	Figure 3	$0 \in \mathbb{Z}_q$	$(q+1)/4$
-1	0	$q < -1$	$L(\|q\|, 1)$	3	Figure 3	$0 \in \mathbb{Z}_{\|q\|}$	$(-\|q\| + 7)/4$
-1	-1	-1	$L(3, -1)$	3	Figure 3	$1 \in \mathbb{Z}_3$	$4/3$
-1	-1	2	$L(3, 1)$	3	Figure 3	$1 \in \mathbb{Z}_3$	$2/3$
-1	-1	$q \leq -2$	$L(-2q+1, -q+1)$	3	Figure 3	$\|q\| + 1 \in \mathbb{Z}_{2\|q\|+1}$	$\frac{-q^2-6q+3}{-4q+2}$
-1	-1	$q \geq 3$	$L(-2q+1, -q+1)$	3	Figure 3	$q - 1 \in \mathbb{Z}_{2q-1}$	$\frac{-q^2}{-4q+2}$

TABLE 4. The case $r = -1$, $|p| < 2$ or $|q| < 2$ (and the case $(p, q, r) = (2, q, -1)$)

In **Table 4**, we need to determine the binding number $bn(\xi)$ for the rows 4, 7, 9, and 11. For the other rows, see Remark 3.1.

• If $p = 2, q \leq -2, r = -1$, then using the corresponding matrix $\mathcal{L}_{2,q,-1}$, we have
$H_1(M) = \langle\, \mu_1, \mu_2, \cdots, \mu_{|q|+3} |\; \mathcal{L}_{2,q,-1}[\mu]^T_{|q|+3} = [\mathbf{0}]^T_{|q|+3} \,\rangle = \langle\, \mu_2 |\; (|q|+2)\mu_2 = 0 \,\rangle \cong \mathbb{Z}_{|q|+2}$, and $\mu_1 = |q|\mu_2$. Therefore,

$$c_1(\xi_{2,q,-1}) = PD^{-1}(\mu_1) = PD^{-1}(|q|\mu_2) = |q| \in \mathbb{Z}_{|q|+2}.$$

Thus, if $q \leq -2$, then $\xi_{2,q,-1}$ is not isotopic to $\eta_{|q-2|}$ as $c_1(\eta_{|q-2|}) = 0$ implying that $bn(\xi_{2,q,-1}) = 3$ by Theorem 1.1.

• If $p = 0, q \leq -1, r = -1$ (the rows 7 or 9), then $c_1(\xi_{0,q,-1}) = 0$ and so we need to compute $d_3(\xi_{0,q,-1})$. Let K_i's be the components of $\mathbb{L}_{0,q,-1}$ as in Figure 12. Then

$$\mathcal{L}_{0,q,-1} = \begin{pmatrix} -1 & -1 & -1 & \cdots & -1 \\ -1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & A_{|q|} & \\ \cdot & \cdot & & & \\ -1 & 0 & & & \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & A_{|q|} & \\ \cdot & \cdot & & & \\ 0 & 0 & & & \end{pmatrix}$$

By diagonalizing the first two rows of $\mathcal{L}_{0,q,-1}$, we obtain the matrix on the right. So $\sigma(\mathcal{L}_{0,q,-1}) = \sigma(A_{|q|}) = |q| - 1$. The contact surgery diagram for $\xi_{0,q,-1}$ and the corresponding 4-manifold $X_{0,q,-1}$ (with $\partial X_{0,q,-1} = M$) are given in Figure 12.

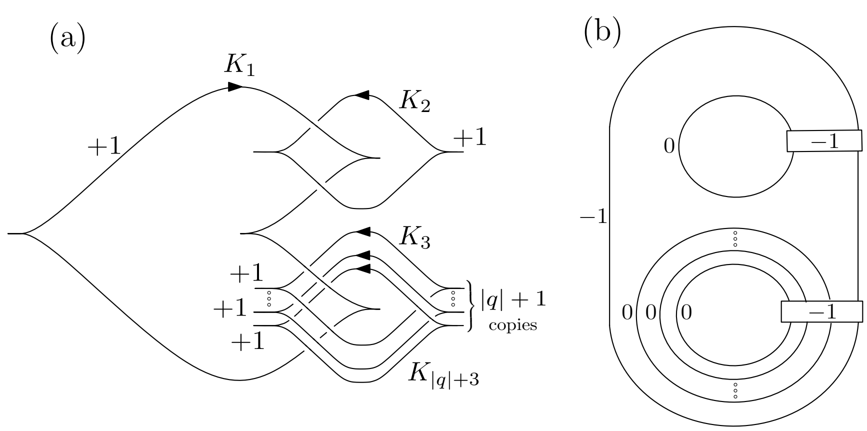


FIGURE 12. (a) The overtwisted contact structure $\xi_{0,q,-1}$ on $L(|q|, 1)$, (b) The corresponding framed link $\mathbb{L}_{0,q,-1}$.

Then the system $\mathcal{L}_{0,q,-1}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2) \cdots rot(K_{|q|+3})]^T = [1\ 0\ 0 \cdots 0]^T$ has the solution $[\mathbf{b}] = [0\ -1\ 0 \cdots 0]$, and so $c^2 = 0$. Moreover, $\chi(X_{0,q,-1}) = |q|+4$ and $s = |q|+3$.

Therefore, we obtain $d_3(\xi_{0,q,-1}) = (-|q|+7)/4$ implying that $\xi_{0,q,-1}$ is not isotopic to $\eta_{|q|}$ as $d_3(\eta_{|q|}) = (-|q|+3)/4$. Hence, $bn(\xi_{0,q,-1}) = 3$ by Theorem 1.1.

• If $p=-1, q=2, r=-1$, we have $c_1(\xi_{-1,2,-1}) = 1$ implying that $bn(\xi_{-1,2,-1}) = 3$. To see this, note that $c_1(\xi{-1,2,-1}) = PD^{-1}(\mu_1)$ where μ_1 is the meridian of the surgery curve corresponding K_1. Then using

$$\mathcal{L}_{-1,2,-1} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

we get $H_1(M) = \langle\ \mu_1, \mu_2, \mu_3, \mu_4 |\ \mathcal{L}_{-1,2,-1}[\mu]_4^T = [\mathbf{0}]_4^T\ \rangle = \langle\ \mu_2 |\ 3\mu_2 = 0\ \rangle \cong \mathbb{Z}_3$, and $\mu_1 = -2\mu_2$. Therefore, we compute

$$c_1(\xi_{-1,2,-1}) = PD^{-1}(\mu_1) = PD^{-1}(-2\mu_2) = -2 \in \mathbb{Z}_3 \equiv 1 \in \mathbb{Z}_3.$$

r	p	q	resulting M	$bn(\xi)$	diagram for ξ	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
2	-1	$q \leq -3$	$L(\|q-2\|, 1)$	3	Figure 3	$2 \in \mathbb{Z}_{\|q-2\|}$	$\frac{-q^2-3q+6}{-4q+8}$
2	-1	$q=-1$	$L(3,1)$	3	Figure 3	$2 \in \mathbb{Z}_3$	$-5/6$
2	-1	$q=-2$	$L(4,1)$	3	Figure 3	$2 \in \mathbb{Z}_4$	$-1/4$
2	-1	1	S^3	2	$(\star)\ m=1$	$0 \in \{0\}$	$1/2$
2	-1	2	$S^1 \times S^2$	3	Figure 3	$2 \in \mathbb{Z}$	$\notin Q$
2	-1	3	S^3	3	Figure 3	$0 \in \{0\}$	$3/2$
2	-1	$q \geq 4$	$L(q-2,-1)$	3	Figure 3	$q-4 \in \mathbb{Z}_{q-2}$	$\frac{q^2-3q+6}{4q-8}$
2	0	0	$S^1 \times S^2 \# L(2,-1)$	3	Figure 3	$0 \in \mathbb{Z} \oplus \mathbb{Z}_2$	$1/4$
2	0	1	$S^3 \# L(2,-1)$	2	$(*)\ m=2$	$0 \in \mathbb{Z}_2$	$-1/4$
2	0	$q>1$	$L(q,-1) \# L(2,-1)$	3	Figure 3	$0 \in \mathbb{Z}_q \oplus \mathbb{Z}_2$	$(q-2)/4$
2	0	$q<0$	$L(\|q\|,1) \# L(2,-1)$	3	Figure 3	$0 \in \mathbb{Z}_{\|q\|} \oplus \mathbb{Z}_2$	$(-\|q\|+4)/4$
2	1	-2	$L(4,-1)$	3	Figure 3	$2 \in \mathbb{Z}_4$	$1/2$
2	1	1	$L(-5,2)$	3	Figure 3	$2 \in \mathbb{Z}_5$	$-1/10$
2	1	$q \leq -3$	$L(-3q-2, q+1)$	3	Figure 3	$2 \in \mathbb{Z}_{-3q-2}$	$\frac{3q^2+15q+10}{12q+8}$
2	1	$q \geq 2$	$L(-3q-2, q+1)$	3	Figure 3	$2 \in \mathbb{Z}_{3q+2}$	$\frac{3q^2-3q-2}{12q+8}$

TABLE 5. The case $r=2$ ($|p| < 2$ or $|q| < 2$).

In **Table 5**, we need to compute the binding number $bn(\xi)$ for the rows 1, 2, and 3. For the other rows, again see Remark 3.1.

For the first three rows in Table 5, the contact structure $\xi_{-1,q,2}$ on $L(|q-2|,1)$ and the link $\mathbb{L}_{-1,q,2}$ ($q \leq -1$) are given in Figure 13. We write the linking matrix $\mathcal{L}_{-1,q,2}$ as the matrix on the left below. It is not hard to see that $c_1(\xi_{-1,q,2}) = 2 \in \mathbb{Z}_{|q-2|}$, and so $bn(\xi_{-1,q,2}) = 3$. As an illustration we will compute $d_3(\xi_{-1,q,2})$ (even though it is not necessary for the proof). The matrix on the right below is obtained by diagonalizing the first two rows of $\mathcal{L}_{-1,q,2}$. So we compute $\sigma(\mathcal{L}_{-1,q,2}) = 2+\sigma(A_1)+\sigma(B_{|q|})$ which is $|q|-1$ if $q \leq -3$, and is equal to 2 if $q=-1,-2$ (recall $\sigma(B_n)$ is $n-3$ if $n \geq 3$, and 0 if $n = 1, 2$).

$$
\mathcal{L}_{-1,q,2} = \left(\begin{array}{cc:cc:cccc}
-3 & -2 & -1 & -1 & -1 & -1 & \cdots & -1 \\
-2 & -3 & -1 & -1 & -1 & -1 & \cdots & -1 \\ \hdashline
-1 & -1 & \multicolumn{2}{c:}{\multirow{2}{*}{A_1}} & 0 & 0 & \cdots & 0 \\
-1 & -1 & & & 0 & 0 & \cdots & 0 \\ \hdashline
-1 & -1 & 0 & 0 & & & & \\
-1 & -1 & 0 & 0 & & & & \\
\vdots & \vdots & \vdots & \vdots & & \multicolumn{2}{c}{A_{|q|}} & \\
-1 & -1 & 0 & 0 & & & &
\end{array}\right)
\xrightarrow{\text{diagonalize}}
\left(\begin{array}{cc:cc:cccc}
2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1/2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hdashline
0 & 0 & \multicolumn{2}{c:}{\multirow{2}{*}{A_1}} & 0 & 0 & \cdots & 0 \\
0 & 0 & & & 0 & 0 & \cdots & 0 \\ \hdashline
0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & & & & \\
\vdots & \vdots & \vdots & \vdots & & \multicolumn{2}{c}{B_{|q|}} & \\
0 & 0 & 0 & 0 & & & &
\end{array}\right)
$$

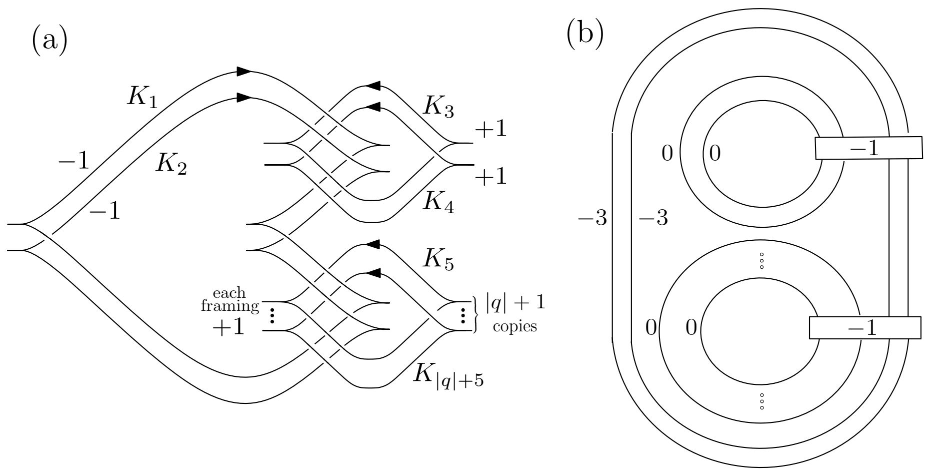


FIGURE 13. (a) The contact structure $\xi_{-1,q,2}$ on $L(|q-2|,1)$ for $q \leq -1$, (b) The corresponding framed link $\mathbb{L}_{-1,q,2}$.

By a standard calculation, the system

$$\mathcal{L}_{-1,q,2}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2) \cdots rot(K_{|q|+5})]^T = [1\ 1\ 0 \cdots 0]^T$$

has the solution $[\mathbf{b}] = [\frac{|q|}{|q|+2}\ \frac{|q|}{|q|+2}\ \frac{-2|q|}{|q|+2}\ \frac{-2|q|}{|q|+2}\ \frac{-2}{|q|+2} \cdots \frac{-2}{|q|+2}]$ for $q \leq -1$, and so we compute

$$c^2 = [\mathbf{b}]\mathcal{L}_{-1,q,2}[\mathbf{b}]^T = 2|q|/(|q|+2).$$

• If $p = -1, q = -1, r = 2$, then $c^2 = 2/3$, $\sigma(X_{-1,-1,2}) = 2$, $\chi(X_{-1,-1,2}) = 7$, and $s = 4$. So we get $d_3(\xi_{-1,-1,2}) = -5/6$.

• If $p = -1, q = -2, r = 2$, then $c^2 = 1$, $\sigma(X_{-1,-2,2}) = 2$, $\chi(X_{-1,-2,2}) = 8$, and $s = 5$. Therefore, we get $d_3(\xi_{-1,-2,2}) = -1/4$.

• If $p = -1, q \leq -3, r = 2$, then $c^2 = 2|q|/(|q|+2)$, $\sigma(X_{-1,q,2}) = |q| - 1$, $\chi(X_{-1,q,2}) = |q| + 6$, and $s = |q| + 3$. So we obtain

$$d_3(\xi_{-1,q,2}) = \frac{-q^2 - 3q + 6}{-4q + 8}.$$

In **Table 6**, we need to compute the binding number $bn(\xi)$ for the rows 7, 9, 10, and 13. For the other rows, see Remark 3.1.

• If $p = 1, q \leq -4, r = -2$, the contact structure $\xi_{1,q,-2}$ on $L(|q+2|, 1)$ and the link $\mathbb{L}_{1,q,-2}$ are given in Figure 14.

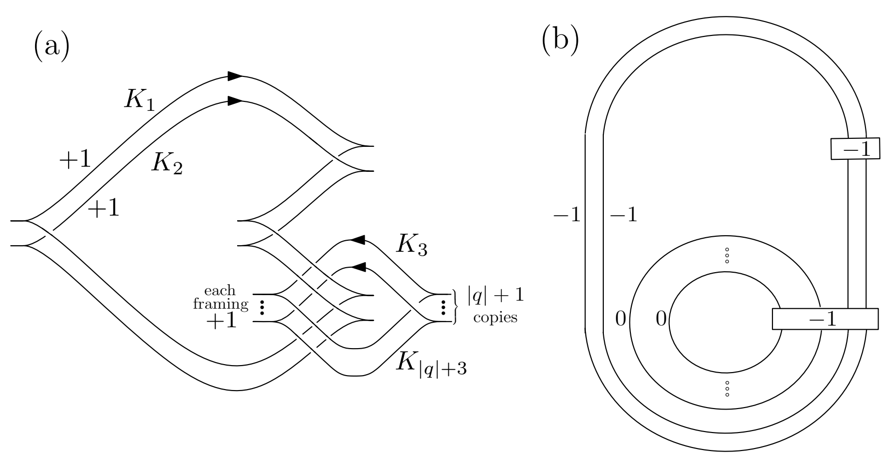


FIGURE 14. (a) The contact structure $\xi_{1,q,-2}$ on $L(|q+2|, 1)$ for $q < -3$, (b) The corresponding framed link $\mathbb{L}_{1,q,-2}$.

r	p	q	resulting M	$bn(\xi)$	diagram for ξ	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
-2	1	$q \geq 3$	$L(q+2,-1)$	3	Figure 3	$2 \in \mathbb{Z}_{q+2}$	$\frac{q^2+q+2}{4q+8}$
-2	1	2	$L(4,-1)$	3	Figure 3	$0 \in \mathbb{Z}_4$	$1/2$
-2	1	1	$L(3,-1)$	3	Figure 3	$2 \in \mathbb{Z}_3$	$1/3$
-2	1	-1	S^3	2	$(\star)$ $m=1$	$0 \in \{0\}$	$1/2$
-2	1	-2	$S^1 \times S^2$	3	Figure 3	$2 \in \mathbb{Z}$	$\notin Q$
-2	1	-3	S^3	3	Figure 3	$0 \in \{0\}$	$-1/2$
-2	1	$q \leq -4$	$L(\|q+2\|,1)$	3	Figure 3	$\|q\|-4 \in \mathbb{Z}_{\|q\|-2}$	$\frac{-q^2-7q-14}{-4q-8}$
-2	0	0	$S^1 \times S^2 \# L(2,1)$	3	Figure 3	$0 \in \mathbb{Z} \oplus \mathbb{Z}_2$	$3/4$
-2	0	1	$S^3 \# L(2,1)$	2	$(\star)$ $m=2$	$0 \in \mathbb{Z}_2$	$1/4$
-2	0	-1	$S^3 \# L(2,1)$	3	Figure 3	$0 \in \mathbb{Z}_2$	$5/4$
-2	0	$q \geq 2$	$L(q,-1) \# L(2,1)$	3	Figure 3	$0 \in \mathbb{Z}_q \oplus \mathbb{Z}_2$	$q/4$
-2	0	$q \leq -2$	$L(\|q\|,1) \# L(2,1)$	3	Figure 3	$0 \in \mathbb{Z}_{\|q\|} \oplus \mathbb{Z}_2$	$(-\|q\|+6)/4$
-2	-1	2	$L(4,1)$	3	Figure 3	$0 \in \mathbb{Z}_4$	$1/2$
-2	-1	-1	$L(-5,-2)$	3	Figure 3	$2 \in \mathbb{Z}_5$	$11/10$
-2	-1	$q \leq -2$	$L(3q-2,q-1)$	3	Figure 3	$2 \in \mathbb{Z}_{3\|q\|+2}$	$\frac{-3q^2-15q+10}{-12q+8}$
-2	-1	$\|q\| \geq 3$	$L(3q-2,q-1)$	3	Figure 3	$2 \in \mathbb{Z}_{3q-2}$	$\frac{-3q^2+3q-2}{-12q+8}$

TABLE 6. The case $r=-2$ ($|p|<2$ or $|q|<2$).

We will first compute that $c_1(\xi_{1,q,-2}) = |q|-4 \in \mathbb{Z}_{|q|-2}$ (so $bn(\xi_{1,q,-2})=3$), and then (even though it is not necessary for the proof) we will evaluate $d_3(\xi_{1,q,-2})$ as an another sample computation. Using $\mathcal{L}_{1,q,-2}$ (on the left below), we have

$$\begin{aligned} H_1(M) &= \langle\, \mu_1,\mu_2,\cdots,\mu_{|q|+3} \,|\, \mathcal{L}_{1,q,-2}[\mu]^T_{|q|+3} = [\mathbf{0}]^T_{|q|+3} \,\rangle \\ &= \langle\, \mu_1,\mu_3 \,|\, -3\mu_1-(|q|+1)\mu_3=0, -2\mu_1-|q|\mu_3=0 \,\rangle \\ &= \langle\, \mu_3 \,|\, (|q|-2)\mu_3=0 \,\rangle \cong \mathbb{Z}_{|q|-2}, \end{aligned}$$

and also we have $\mu_1=\mu_2=-\mu_3$. Therefore, we obtain

$$c_1(\xi_{2,q,-1}) = PD^{-1}(\mu_1+\mu_2) = PD^{-1}(-2\mu_3) = -2 \equiv |q|-4 \in \mathbb{Z}_{|q|-2}.$$

The matrix on the right below is obtained by diagonalizing the first two rows of $\mathcal{L}_{1,q,-2}$. So we compute $\sigma(\mathcal{L}_{1,q,-2}) = 0+\sigma(C_{|q|}) = |q|-1$ (recall $\sigma(C_n)=n-1$ if $n \geq 2$).

$$\mathcal{L}_{1,q,-2} = \left[\begin{array}{cc|ccc} -1 & -2 & -1 & \cdots & -1 \\ -2 & -1 & -1 & \cdots & -1 \\ \hline -1 & -1 & & & \\ \vdots & \vdots & & A_{|q|} & \\ -1 & -1 & & & \end{array}\right] \xrightarrow{\text{diagonalize}} \left[\begin{array}{cc|ccc} 2 & 0 & 0 & \cdots & 0 \\ 0 & -1/2 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & C_{|q|} & \\ 0 & 0 & & & \end{array}\right]$$

By a standard calculation, the system

$$\mathcal{L}_{1,q,-2}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2) \cdots rot(K_{|q|+3})]^T = [1\ 1\ 0 \cdots 0]^T$$

has the solution $[\mathbf{b}] = [\frac{-|q|}{|q|-2}\ \frac{-|q|}{|q|-2}\ \frac{2}{|q|-2}\ \cdots\ \frac{2}{|q|-2}]$, and so we obtain

$$c^2 = [\mathbf{b}]\mathcal{L}_{1,q,-2}[\mathbf{b}]^T = -2|q|/(|q|-2).$$

Moreover, $\chi(X_{1,q,-2}) = |q|+4$, and $s = |q|+3$. So we compute

$$d_3(\xi_{1,q,-2}) = \frac{-q^2-7q-14}{-4q-8}.$$

- If $p=0, q=1, r=-2$, then $\xi_{0,1,-2}$ and $\mathbb{L}_{0,1,-2}$ are given in Figure 15.

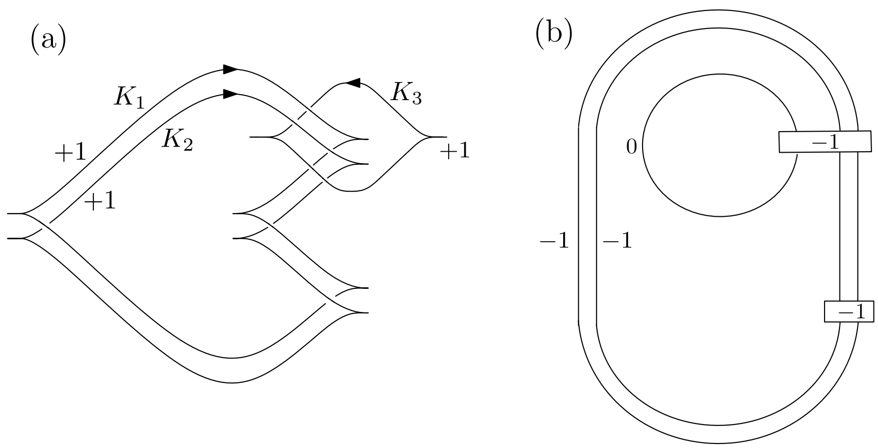


Figure 15. (a) The contact structure $\xi_{0,1,-2}$ on $S^3 \# L(2,1) \approx L(2,1)$, (b) The corresponding framed link $\mathbb{L}_{0,1,-2}$.

One can get $c_1(\xi_{0,1,-2}) = 0$, so we need $d_3(\xi_{0,1,-2})$. The corresponding linking matrix is

$$\mathcal{L}_{0,1,-2} = \begin{pmatrix} -1 & -2 & -1 \\ -2 & -1 & -1 \\ -1 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We diagonalize $\mathcal{L}_{0,1,-2}$, and obtain the matrix on the right. So $\sigma(\mathcal{L}_{0,1,-2}) = 1$. We find that the system $\mathcal{L}_{0,1,-2}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2)\ rot(K_3)]^T = [1\ 1\ 0]^T$ has the solution $[\mathbf{b}] = [0\ 0\ -1]$, and so $c^2 = 0$. Also we have $\chi(X_{0,1,-2}) = 4$ and $s = 3$. So we get $d_3(\xi_{0,1,-2}) = 1/4 = d_3(\eta_2)$ which implies that $\xi_{0,1,-2}$ is isotopic to η_2. Thus, $bn(\xi_{0,1,-2}) = 2$ by Theorem 1.1.

• If $p = 0, q = -1, r = -2$, then the contact structure $\xi_{0,1,-2}$ on $L(2,1)$ and the link $\mathbb{L}_{0,-1,-2}$ describing $X_{0,-1,-2}$ are given in Figure 16. It is easy to check $c_1(\xi_{0,-1,-2}) = 0$, so we compute $d_3(\xi_{0,-1,-2})$:

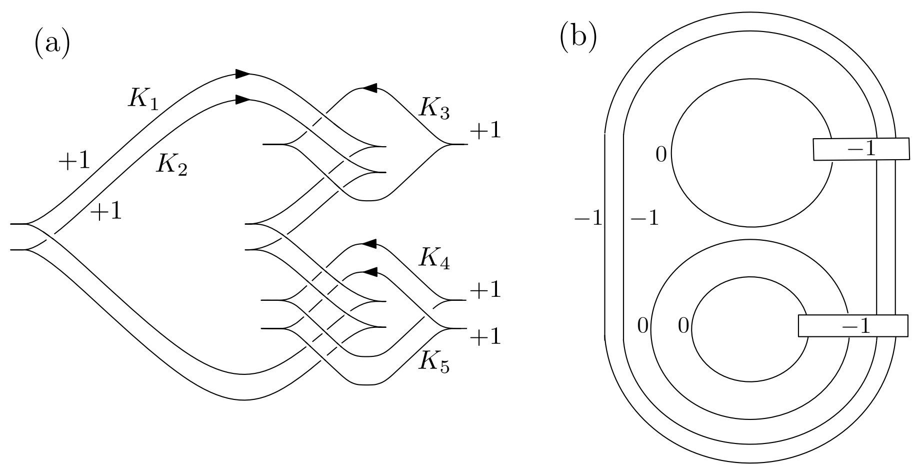


FIGURE 16. (a) The contact structure $\xi_{0,-1,-2}$ on $S^3\#L(2,1) \approx L(2,1)$, (b) The corresponding framed link $\mathbb{L}_{0,-1,-2}$.

The corresponding linking matrix is

$$\mathcal{L}_{0,-1,-2} = \begin{pmatrix} -1 & -2 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

We diagonalize $\mathcal{L}_{0,-1,-2}$, and obtain the matrix on the right. So $\sigma(\mathcal{L}_{0,-1,-2}) = 1$. The system

$$\mathcal{L}_{0,-1,-2}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2)\ rot(K_3)\ rot(K_4)\ rot(K_5)]^T = [1\ \ 1\ \ 0\ \ 0\ \ 0]^T$$

has the solution $[\mathbf{b}] = [0\ \ 0\ -1\ \ 0\ \ 0]$ which yields $c^2 = 0$. Also we have $\chi(X_{0,-1,-2}) = 4$ and $s = 3$. So we get $d_3(\xi_{0,-1,-2}) = 5/4 \neq 1/4 = d_3(\eta_2)$. Therefore, $\xi_{0,-1,-2}$ is not isotopic to η_2, and so $bn(\xi_{0,-1,-2}) = 3$ by Theorem 1.1.

• If $p = -1, q = 2, r = -2$, then the contact structure $\xi_{-1,2,-2}$ on $L(4,1)$ and the link $\mathbb{L}_{-1,2,-2}$ describing $X_{-1,2,-2}$ are given in Figure 17. We compute that $c_1(\xi_{-1,2,-2}) = 0$, so we need to find $d_3(\xi_{-1,2,-2})$.

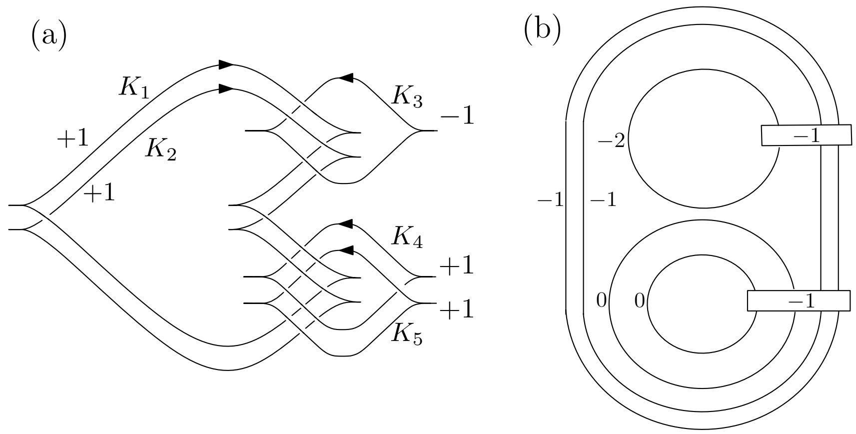


FIGURE 17. (a) The contact structure $\xi_{-1,2,-2}$ on $L(4,1)$,
(b) The corresponding framed link $\mathbb{L}_{-1,2,-2}$.

The corresponding linking matrix is

$$\mathcal{L}_{-1,2,-2} = \begin{pmatrix} -1 & -2 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 \\ -1 & -1 & -2 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 \\ -1 & -1 & 0 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

We diagonalize $\mathcal{L}_{-1,2,-2}$, and obtain the matrix on the right. So $\sigma(\mathcal{L}_{-1,2,-2}) = 1$. The system

$$\mathcal{L}_{-1,2,-2}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2)\ rot(K_3)\ rot(K_4)\ rot(K_5)]^T = [1\ \ 1\ \ 0\ \ 0\ \ 0]^T$$

has the solution $[\mathbf{b}] = [1/2 \;\; 1/2 \;\; -1/2 \;\; -1 \;\; -1]$, so we compute $c^2 = 1$. Moreover, $\chi(X_{-1,2,-2}) = 6$ and $s = 4$. Then we get $d_3(\xi_{-1,2,-2}) = 1/2 \neq -1/4 = d_3(\eta_4)$. Therefore, $\xi_{-1,2,-2}$ is not isotopic to η_4, and so $bn(\xi_{-1,2,-2}) = 3$ by Theorem 1.1.

p	q	resulting M	$bn(\xi)$	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
-1	0	$L(r,-1)$	3	$0 \in \mathbb{Z}_r$	$(r+1)/4$
-1	1	S^3	2	$0 \in \{0\}$	$1/2$
-1	2	$L(r-2,-1)$	3	$2 \in \mathbb{Z}_{r-2}$	$(r^2-3r+6)/(4r-8)$
-1	$q \leq -1$	$L(qr-q-r,-q+1)$	3	$r \in \mathbb{Z}_{\lvert q\rvert r-\lvert q\rvert+r}$	$\frac{q^2r+qr^2-q^2-r^2-q-r}{4qr-4q-4r}$
-1	$q \geq 3$	$L(qr-q-r,-q+1)$	3	$r \in \mathbb{Z}_{qr-q-r}$	$\frac{q^2r+qr^2-q^2-r^2-6qr+5q+5r}{4qr-4q-4r}$
0	0	$S^1 \times S^2 \# L(r,-1)$	3	$0 \in \mathbb{Z} \oplus \mathbb{Z}_r$	$(r-1)/4$
0	1	$S^3 \# L(r,-1)$	2	$0 \in \mathbb{Z}_r$	$(-r+3)/4$
0	$q \leq -2$	$L(\lvert q\rvert,1) \# L(r,-1)$	3	$0 \in \mathbb{Z}_{\lvert q\rvert} \oplus \mathbb{Z}_r$	$(q+r+2)/4$
0	$q \geq 2$	$L(q,-1) \# L(r,-1)$	3	$0 \in \mathbb{Z}_q \oplus \mathbb{Z}_r$	$(q+r-4)/4$
1	-2	$L(r+2,-1)$	3	$r \in \mathbb{Z}_{r+2}$	$(r^2+r+2)/(4r+8)$
1	1	$L(2r+1,-2)$	3	$r \in \mathbb{Z}_{2r+1}$	$(r^2-2r-1)/(4r+2)$
1	$q \leq -3$	$L(qr+q+r,-q-1)$	3	$r \in \mathbb{Z}_{\lvert q\rvert r+\lvert q\rvert-r}$	$\frac{q^2r+qr^2+q^2+r^2+4qr+3q+3r}{4qr+4q+4r}$
1	$q \geq 2$	$L(qr+q+r,-q-1)$	3	$r \in \mathbb{Z}_{qr+q+r}$	$\frac{q^2r+qr^2+q^2+r^2-2qr-3q-3r}{4qr+4q+4r}$

Table 7. The case $r > 2$ ($\lvert p\rvert < 2$ or $\lvert q\rvert < 2$).

In **Table 7**, we do not need any computation to find $bn(\xi)$: For any row, we can use Remark 3.1. For example, in the 1^{st} row, we have an overtwisted contact structure on the lens space $L(m,-1)$ for some $m \geq 1$. Therefore, the resulting contact manifold is not listed in Theorem 1.1, and hence we must have $bn(\xi) = 3$.

In **Table 8**, we need to compute the binding number $bn(\xi)$ for the rows 1, 3, 7, and 10. For the other rows, see Remark 3.1.

• If $p = 1, q = -2, r < -2$, $\xi_{1,-2,r}$ is an overtwisted contact structure on $L(\lvert r+2\rvert, 1)$. It is not hard to see that $c_1(\xi_{1,-2,r}) = 2 \in \mathbb{Z}_{\lvert r\rvert-2}$. Therefore, we immediately get $bn(\xi_{1,q,-2}) = 3$ because $c_1(\eta_{\lvert r+2\rvert}) = 0$.

p	q	resulting M	$bn(\xi)$	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
1	-2	$L(\lvert r+2\rvert,1)$	3	$2 \in \mathbb{Z}_{\lvert r\rvert-2}$	$(r^2+7r+14)/(4r+8)$
1	-1	S^3	2	$0 \in \{0\}$	$1/2$
1	0	$L(\lvert r\rvert,1)$	2	$0 \in \mathbb{Z}_{\lvert r\rvert}$	$(-\lvert r\rvert+3)/4$
1	$q \leq -3$	$L(qr+q+r,-q-1)$	3	$\lvert r\rvert \in \mathbb{Z}_{\lvert q\rvert\lvert r\rvert-\lvert q\rvert-\lvert r\rvert}$	$\frac{q^2r+qr^2+q^2+r^2+10qr+9q+9r}{4qr+4q+4r}$
1	$q \geq 1$	$L(qr+q+r,-q-1)$	3	$\lvert r\rvert \in \mathbb{Z}_{q\lvert r\rvert-q+\lvert r\rvert}$	$\frac{q^2r+qr^2+q^2+r^2+4qr+3q+3r}{4qr+4q+4r}$
0	0	$S^1\times S^2\#L(\lvert r\rvert,1)$	3	$0 \in \mathbb{Z}\oplus\mathbb{Z}_{\lvert r\rvert}$	$(-\lvert r\rvert+5)/4$
0	-1	$S^3\#L(\lvert r\rvert,1)$	3	$0 \in \mathbb{Z}_{\lvert r\rvert}$	$(-\lvert r\rvert+7)/4$
0	$q \leq -2$	$L(\lvert q\rvert,1)\#L(\lvert r\rvert,1)$	3	$0 \in \mathbb{Z}_{\lvert q\rvert}\oplus\mathbb{Z}_{\lvert r\rvert}$	$(q+r+8)/4$
0	$q \geq 2$	$L(q,-1)\#L(\lvert r\rvert,1)$	3	$0 \in \mathbb{Z}_q\oplus\mathbb{Z}_{\lvert r\rvert}$	$(q+r+2)/4$
-1	2	$L(\lvert r-2\rvert,1)$	3	$\lvert r\rvert \in \mathbb{Z}_{\lvert r\rvert+2}$	$(-r^2-3r+6)/(-4r+8)$
-1	-1	$L(-2r+1,2)$	3	$\lvert r\rvert \in \mathbb{Z}_{2\lvert r\rvert+1}$	$(-r^2-6r+3)/(-4r+2)$
-1	$q \leq -2$	$L(qr-q-r,-q+1)$	3	$\lvert r\rvert \in \mathbb{Z}_{\lvert q\rvert\lvert r\rvert+\lvert q\rvert+\lvert r\rvert}$	$\frac{q^2r+qr^2-q^2-r^2+6qr-7q-7r}{4qr-4q-4r}$
-1	$q \geq 3$	$L(qr-q-r,-q+1)$	3	$\lvert r\rvert \in \mathbb{Z}_{q\lvert r\rvert+q-\lvert r\rvert}$	$\frac{q^2r+qr^2-q^2-r^2-q-r}{4qr-4q-4r}$

TABLE 8. The case $r < -2$ ($|p| < 2$ or $|q| < 2$).

• If $p = 1, q = 0, r < -2$, the contact structure $\xi_{1,0,r}$ on $L(|r|,1)$ and the link $\mathbb{L}_{1,0,r}$ are given in Figure 18. It is easy to see that $c_1(\xi_{1,0,r}) = 0 \in \mathbb{Z}_{|r|}$, so we need $d_3(\xi_{1,0,r})$: The corresponding linking matrix is on the left below. Diagonalize $\mathcal{L}_{1,0,r}$ to get the matrix on the right. Therefore, $\sigma(\mathcal{L}_{1,0,r}) = |r| - 1$.

$$\mathcal{L}_{1,0,r} = \left(\begin{array}{cccccc:c} -1 & -2 & -2 & \cdots & & -2 & -1 \\ -2 & -1 & -2 & \cdots & & -2 & -1 \\ -2 & -2 & -1 & & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & & -1 & -2 & \\ -2 & -2 & \cdots & & -2 & -1 & -1 \\ \hdashline -1 & -1 & \cdots & & & -1 & 0 \end{array}\right) \xrightarrow{\text{diagonalize}} \left(\begin{array}{cccccc:c} 1 & 0 & 0 & \cdots & & 0 & 0 \\ 0 & 2 & 0 & \cdots & & 0 & 0 \\ 0 & 0 & 2 & & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & & 2 & 0 & \\ 0 & 0 & \cdots & & 0 & 1/2 & 0 \\ \hdashline 0 & 0 & \cdots & & & 0 & -3 \end{array}\right)$$

The system

$$\mathcal{L}_{1,0,r}[\mathbf{b}]^T = [rot(K_1)\ rot(K_2) \cdots rot(K_{|q|+1})]^T = [1 \cdots 1\ 0]^T$$

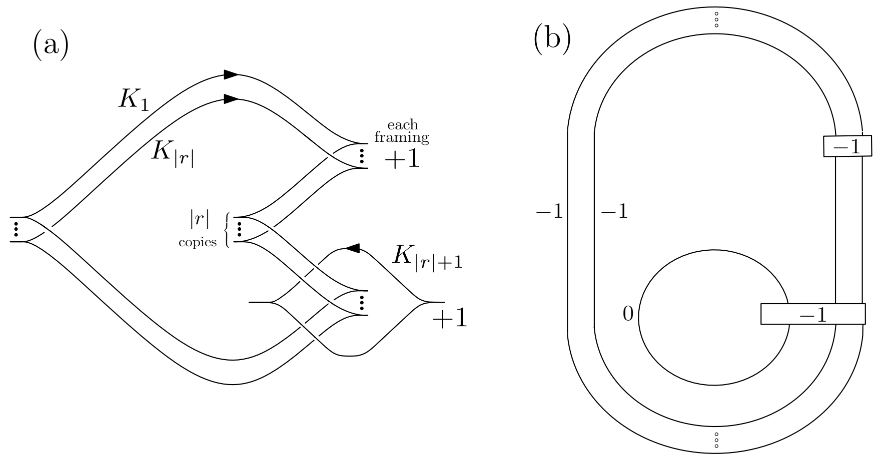


FIGURE 18. (a) The contact structure $\xi_{1,0,r}$ on $L(|r|,1)$ for $r < -2$, (b) The corresponding framed link $\mathbb{L}_{1,0,r}$.

has the solution $[\mathbf{b}] = [0 \cdots 0 \ -1]$, and so we obtain $c^2 = 0$. Moreover, $\chi(X_{1,0,r}) = |r|+2$, and $s = |r| + 1$. So we compute $d_3(\xi_{1,0,r}) = (-|r| + 3)/4 = d_3(\eta_{|r|})$ which implies that $\xi_{1,0,r}$ is isotopic to $\eta_{|r|}$ on $L(|r|,1)$. Thus, $bn(\xi_{1,0,r}) = 2$ by Theorem 1.1.

• If $p = 0, q = -1, r < -2$, the contact structure $\xi_{0,-1,r}$ on $L(|r|,1)$ and the link $\mathbb{L}_{0,-1,r}$ are given in Figure 19. Again we have $c_1(\xi_{0,-1,r}) = 0 \in \mathbb{Z}_{|r|}$, so we need to find $d_3(\xi_{0,-1,r})$: We diagonalize $\mathcal{L}_{0,-1,r}$ and get the matrix on the right below. So, we conclude that $\sigma(\mathcal{L}_{0,-1,r}) = |r| - 1$.

$$\mathcal{L}_{0,-1,r} = \left(\begin{array}{cccccc:ccc}
-1 & -2 & -2 & \cdots & & -2 & -1 & -1 & -1 \\
-2 & -1 & -2 & \cdots & & -2 & -1 & -1 & -1 \\
-2 & -2 & -1 & & & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & & & -1 & -2 & & & \\
-2 & -2 & \cdots & & -2 & -1 & -1 & -1 & -1 \\
\hdashline
-1 & -1 & \cdots & & & -1 & 0 & 0 & 0 \\
-1 & -1 & \cdots & & & -1 & 0 & 0 & -1 \\
-1 & -1 & \cdots & & & -1 & 0 & -1 & 0
\end{array}\right) \xrightarrow{\text{diagonalize}} \left(\begin{array}{cccccc:ccc}
1 & 0 & 0 & \cdots & & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & \cdots & & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & & & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & & \ddots & & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & & & 2 & 0 & & & \\
0 & 0 & \cdots & & 0 & 1/2 & 0 & 0 & 0 \\
\hdashline
0 & 0 & \cdots & & & 0 & -3 & 0 & 0 \\
0 & 0 & \cdots & & & 0 & 0 & 2 & 0 \\
0 & 0 & \cdots & & & 0 & 0 & 0 & -1/2
\end{array}\right)$$

The system

$$\mathcal{L}_{0,-1,r}[\mathbf{b}]^T = [rot(K_1) \ rot(K_2) \cdots rot(K_{|q|+3})]^T = [1 \cdots 1 \ 0 \ 0 \ 0]^T$$

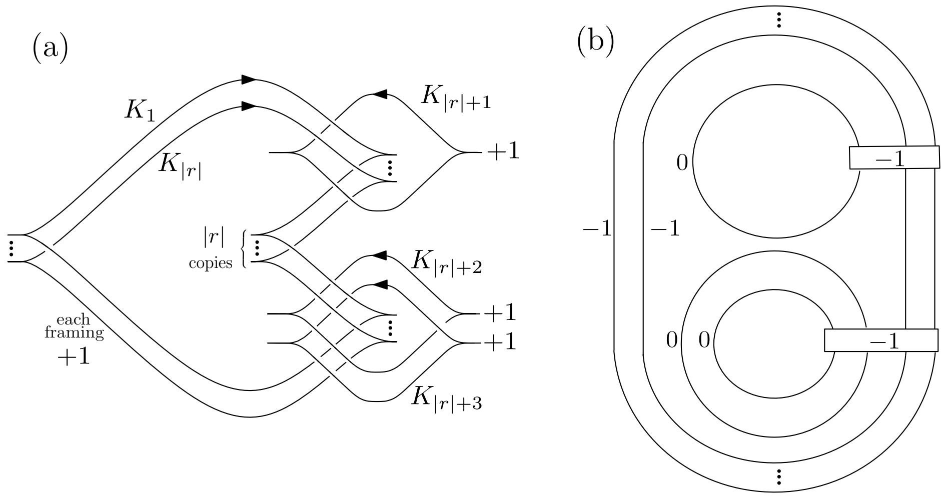


FIGURE 19. (a) The contact structure $\xi_{0,-1,r}$ on $L(|r|,1)$ for $r<-2$, (b) The corresponding framed link $\mathbb{L}_{0,-1,r}$.

has the solution $[\mathbf{b}] = [0 \cdots\ 0\ 1\ 0\ 0]$, so we get $c^2 = 0$. Also $\chi(X_{1,q,-2}) = |r| + 4$, and $s = |r| + 3$. So we compute $d_3(\xi_{0,-1,r}) = (-|r| + 7)/4 \neq (-|r| + 3)/4 = d_3(\eta_{|r|})$ implying that $\xi_{0,-1,r} \ncong \eta_{|r|}$ on $L(|r|, 1)$. Hence, $bn(\xi_{0,-1,r}) = 3$ by Theorem 1.1.

• If $p = -1, q = 2, r < -2$, we have $bn(\xi_{-1,2,r}) = 3$ because $c_1(\xi_{-1,2,r}) = |r| \in \mathbb{Z}_{|r|+2}$. We compute $c_1(\xi_{-1,2,r})$ as follows: We use the linking matrix $\mathcal{L}_{-1,2,r}$ to get the representation

$$\begin{aligned} H_1(M) &= \langle\, \mu_1, \mu_2, \cdots, \mu_{|r|+3} \,|\, \mathcal{L}_{-1,2,r}[\mu]^T_{|r|+3} = [\mathbf{0}]^T_{|r|+3} \,\rangle \\ &= \langle\, \mu_1 \,|\, (|r|+2)\mu_1 = 0 \,\rangle \cong \mathbb{Z}_{|r|+2}. \end{aligned}$$

Moreover, using the relations given by $\mathcal{L}_{-1,2,r}$ we have $\mu_1 = \mu_2 \cdots = \mu_{|r|}$ (μ_i's are the meridians as before). Therefore, we obtain

$$c_1(\xi_{-1,2,r}) = PD^{-1}(\mu_1 + \cdots + \mu_{|r|}) = PD^{-1}(|r|\mu_1) = |r| \in \mathbb{Z}_{|r|+2}.$$

To finish the proof, in each table above we find each particular case for (p, q, r) such that the corresponding contact structure $\xi_{p,q,r}$ has binding number 2. Note that the conditions on p, q, r given in the statement of the theorem excludes exactly these cases. This completes the proof. □

4. Remarks on the remaining cases

Assume that $r = 0, \pm 1, |p| \geq 2, |q| \geq 2$. We list all possible contact structures in Table 9. These are the only remaining cases from which we still get lens spaces or their connected sums. Notice that we have already considered the cases $(-2, q, 1)$, and $(2, q, -1)$ in Tables 3 and 4, so we do not list them here.

r	p	q	resulting M	$c_1(\xi) \in H^2(M)$	$d_3(\xi)$
0	$p \geq 2$	$q \geq 2$	$L(p,-1)\#L(q,-1)$	$0 \in \mathbb{Z}_p \oplus \mathbb{Z}_q$	$(p+q-4)/4$
0	$p \geq 2$	$q \leq -2$	$L(p,-1)\#L(\lvert q\rvert,1)$	$0 \in \mathbb{Z}_p \oplus \mathbb{Z}_{\lvert q\rvert}$	$(p+q+2)/4$
0	$p \leq -2$	$q \leq -2$	$L(\lvert p\rvert,1)\#L(\lvert q\rvert,1)$	$0 \in \mathbb{Z}_{\lvert p\rvert} \oplus \mathbb{Z}_{\lvert q\rvert}$	$(p+q+8)/4$
1	$p \geq 2$	$q \geq 2$	$L(pq+p+q,-q-1)$	$-p \in \mathbb{Z}_{pq+p+q}$	$\frac{p^2q+pq^2+p^2+q^2-2pq-3p-3q}{4pq+4p+4q}$
1	$p \geq 2$	$q < -2$	$L(pq+p+q,-q-1)$	$-p \in \mathbb{Z}_{p\lvert q\rvert-p+\lvert q\rvert}$	$\frac{-p^2q+pq^2+p^2+q^2+4pq+3p+3q}{4pq+4p+4q}$
1	$p < -2$	$q < -2$	$L(pq+p+q,-q-1)$	$p \in \mathbb{Z}_{\lvert p\rvert\lvert q\rvert+\lvert p\rvert+\lvert q\rvert}$	$\frac{p^2q+pq^2+p^2+q^2+10pq+9p+9q}{4pq+4p+4q}$
-1	$p > 2$	$q > 2$	$L(pq-p-q,-q+1)$	$-p \in \mathbb{Z}_{pq-p-q}$	$\frac{p^2q+pq^2-p^2-q^2-6pq+5p+5q}{4pq-4p-4q}$
-1	$p > 2$	$q \leq -2$	$L(pq-p-q,-q+1)$	$-p \in \mathbb{Z}_{p\lvert q\rvert+p-\lvert q\rvert}$	$\frac{p^2q+pq^2-p^2-q^2-p-q}{4pq-4p-4q}$
-1	$p \leq -2$	$q \leq -2$	$L(pq-p-q,-q+1)$	$p \in \mathbb{Z}_{\lvert p\rvert\lvert q\rvert+\lvert p\rvert+\lvert q\rvert}$	$\frac{p^2q+pq^2-p^2-q^2+6pq-7p-7q}{4pq-4p-4q}$

TABLE 9. The case $r = 0, \pm 1$, $|p| \geq 2$, $|q| \geq 2$ ($bn(\xi) = 3$ in each row).

As we remarked in Section 1 (after Theorem 1.3) that one can obtain the complete list without any repetition: We first simply find all distinct homeomorphism types of the manifolds which we found in Table 2 through Table 9. Then on a fixed homeomorphism type we compare the pairs (c_1, d_3) coming from the tables to distinguish the contact structures.

Suppose now that M is a prime Seifert fibered manifold which is not a lens space. Then as we remarked before we have $|p| \geq 2, |q| \geq 2$, and $|r| \geq 2$. Then two such triples $(p, q, r), (p', q', r')$ give the same Seifert manifold Y if and only if

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{p'} + \frac{1}{q'} + \frac{1}{r'},$$

and (p', q', r') is a permutation of (p, q, r) (see [JN], for instance). Notice that we can drop the first condition in our case. Switching p and q does not change the contact manifold as we mentioned before. On the other hand, if we switch r and p (or r and q), we might have different contact structures on the same underlying topological manifold.

Another issue is that there are some cases where the first homology group $H_1(Y(p, q, r))$ is not finite. Indeed, consider the linking matrix $\mathcal{L}$ of the surgery diagram given on the right in Figure 2 as below.

$$\mathcal{L} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & p & 0 & 0 \\ 1 & 0 & q & 0 \\ 1 & 0 & 0 & r \end{pmatrix}$$

The determinant $\det(\mathcal{L}) = -r(p+q) - pq = 0$ implies that $r = -\frac{pq}{p+q}$. Thus, if $r \neq -\frac{pq}{p+q}$, then $H_1(Y(p,q,r))$ is finite, and so $d_3(\xi_{p,q,r})$ is still computable since $c_1(\xi_{p,q,r})$ is torsion. For instance, if $p \geq 2, q \geq 2, r \geq 2$ or $p \leq -2, q \leq -2, r \leq -2$, than $\det(\mathcal{L}) \neq 0$, and so we can distinguish the corresponding $\xi_{p,q,r}$ by computing the pair (c_1, d_3). Whereas if the sign of the one of p, q, r is different than the others', then we might have $\det(\mathcal{L}) = 0$. For instance, for the triples $(4, 4, -2), (3, 6, -2)$ and each nonzero integer multiples of them, $\det(A) = 0$. So more care is needed for these cases.

We would like to end the article by a sample computation. Assume that $\det(\mathcal{L}) \neq 0$, and that $r \leq 2, p \geq 2, q \leq 2$ (similar calculations apply for the other cases). We compute the first homology of $M \approx Y(p,q,r)$ as

$$\begin{array}{rcl} H_1(M) & = & \langle\, \mu_1, \mu_2, \cdots, \mu_{p+q+|r|} \,|\, \mathcal{L}_{p,q,r}[\mu]^T_{p+q+|r|} = [\mathbf{0}]^T_{p+q+|r|} \,\rangle \\ & = & \langle\, \mu_1, \mu_{|r|+1}, \mu_{p+|r|} \,|\, R_1, R_2, R_3 \,\rangle \end{array}$$

where the relations of the presentation are

$$\begin{array}{lccccccc} R_1: & -(2|r|-1)\mu_1 & - & (p-1)\mu_{|r|+1} & - & (|q|+1)\mu_{p+|r|} & = & 0 \\ R_2: & & & p\,\mu_{|r|+1} & - & |q|\,\mu_{p+|r|} & = & 0 \\ R_3: & -|r|\,\mu_1 & - & p\,\mu_{|r|+1} & & & = & 0 \end{array}$$

While getting these relations, we also see that $\mu_1 = \mu_2 \cdots \mu_{|r|}$ (recall μ_i's are the meridians to the surgery curves in the family corresponding to r for $i = 1, \cdots, |r|$). Then using this presentation, and knowing that $c_1(\xi_{p,q,r}) = PD^{-1}(|r|\mu_1)$, we can evaluate (understand) $c_1(\xi_{p,q,r})$ in $H^2(M) \cong H_1(M)$.

Now if $c_1(\xi_{p,q,r}) \in H^2(M)$ is a torsion class, then we can also compute $d_3(\xi_{p,q,r})$ as follows: By solving the corresponding linear system we get

$$c^2 = \frac{p|q||r|}{p|q| + p|r| - |q||r|}.$$

Moreover, we compute $\sigma(X_{p,q,r}) = \sigma(\mathcal{L}_{p,q,r}) = -p + |q| + |r|, \chi(X_{p,q,r}) = p + |q| + |r| + 1$, and $s = |q| + |r| + 1$. Hence, using Corollary 2.8, we obtain

$$d_3(\xi_{p,q,r}) = \frac{8pqr + p^2q + p^2r + 4pq^2 + 4qr^2 - pr^2 - q^2r - pq - pr - qr}{4pq + 4pr + 4qr}.$$

References

[AO] S. Akbulut and B. Ozbagci, *Lefschetz fibrations on compact Stein surfaces*, Geom. Topol. **5** (2001), 319–334 (electronic).

[Ar] M. F. Arikan, *On the support genus of a contact structure*, Journal of GGT, **1** (2007) 92-115.

[Bi] J. S. Birman, *Braids, links and mapping class groups*, Annals of Mathematics Studies, No. 82. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974.

[LP] A. Loi and R. Piergallini, *Compact Stein surfaces with boundary as branched covers of B^4,* Invent. Math. **143** (2001), 325–348.

[DG] F. Ding and H. Geiges, *A Legendrian surgery presentation of contact* 3*-manifolds*, Math. Proc. Cambridge Philos. Soc. 136 (2004), no. 3, 583–598.

[DGS] F. Ding, H. Geiges and A. I. Stipsicz, *Surgery diagrams for contact 3-manifolds*, Turkish J. Math, **28** (2004) no. 1, 41–74.

[Et1] J. B. Etnyre, *Planar open book decompositions and contact structures*, Int. Math. Res. Not. 2004, no.**79**, 4255–4267.

[Et2] J. B. Etnyre, *Lectures on open book decompositions and contact structures*, Floer homology, gauge theory, and low-dimensional topology, 103–141, Clay Math. Proc., 5, Amer. Math. Soc., Providence, RI, 2006.

[Et3] J. Etnyre, *Introductory Lectures on Contact Geometry*, Topology and geometry of manifolds (Athens, GA, 2001), 81–107, Proc. Sympos. Pure Math., 71, Amer. Math. Soc., Providence, RI, 2003.

[EG] Y. Eliashberg and M. Gromov *Convex symplectic manifolds*, Proc. of Symp. in Pure Math. **57** (1991), 135–162.

[EO] J. Etnyre and B. Ozbagci, *Invariants of Contact Structures from Open Books*, arXiv:math.GT/0605441, preprint 2006.

[Ge] H. Geiges, *Contact geometry*, Handbook of Differential Geometry. Vol. II, 315–382, Elsevier/North-Holland, Amsterdam, 2006.

[Gd] N Goodman, *Contact Structures and Open Books*, PhD thesis, University of Texas at Austin (2003)

[Gm] R. E. Gompf, *Handlebody construction of Stein surfaces,* Ann. of Math. **148** (1998), 619–693.

[Gi] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, Proceedings of the ICM, Beijing 2002, vol. 2, 405–414.

[Ho] K. Honda, *On the classification of tight contact structures -I*, Geom. Topol. **4** (2000), 309–368 (electronic).

[HKM] K. Honda, W. Kazez and G. Matić, *Right-veering diffeomorphisms of compact surfaces with boundary*, Invent. Math. 169 (2007), no. 2, 427–449.

[JN] M. Jankins and W. D. Neumann, Lectures on Seifert manifolds, Brandeis Lecture Notes 2, Brandeis University (1981).

[Or] P. Orlik, *Seifert Manifolds*, Lecture Notes in Mathematics (1972), Springer-Verlag, Berlin·Heidelberg·New York.

[OS] B. Ozbagci and A. I. Stipsicz, *Surgery on contact 3-manifolds and Stein surfaces*, Bolyai Society Mathematical Studies, **13** (2004), Springer-Verlag, Berlin.

[TW] W. P. Thurston and H. E. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. **52** (1975), 345–347.

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